A notion of multivariate Value at Risk from a directional perspective

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1. INTRODUCTION

2. DIRECTIONAL MULTIVARIATE VALUE AT RISK (MVaR)

3. MARGINAL VaR VS. MVaR

4. COPULAS AND $\text{VaR}_\alpha(X)$

5. NON-PARAMETRIC ESTIMATION

6. ROBUSTNESS

7. CONCLUSIONS
INTRODUCTION

DIRECTIONAL MULTIVARIATE VALUE AT RISK (MVaR)

MARGINAL VaR VS. MVaR

COPULAS AND $VaR^u_\alpha (X)$

NON-PARAMETRIC ESTIMATION

ROBUSTNESS

CONCLUSIONS
Let $X$ be a random variable representing loss, $F$ its distribution function and $0 \leq \alpha \leq 1$. Then,

$$VaR_\alpha(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$
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The *VaR* has became in a benchmark for risk management.
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The VaR has been criticized by Artzner et al. (1999) since it does not encourage diversification.

But defended by Heyde et al. (2009) for its robustness and recently by Daníelsson et al. (2013) for its tail subadditivity.
But, what is one of the problems with this measure?

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But, what is one of the problems with this measure?

It is its extension to the multivariate setting, where

- There is not a unique definition of a multivariate quantile.
- There are a lot of assets in a portfolio. (High Dimension)
- There is dependence among them.
An initial idea to study risk measures related to portfolios

\[ X = (X_1, \ldots, X_n), \]

is to consider a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and then:

- The \textit{VaR} of the joint portfolio is the univariate-one associated to \( f(X) \).
An initial idea to study risk measures related to portfolios

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is to consider a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and then:

- The \( \text{VaR} \) of the joint portfolio is the univariate-one associated to \( f(X) \).
- In Burgert and Rüschendorf (2006),

\[
\begin{align*}
  f(X) &= \sum_{i=1}^{n} X_i \quad \text{or} \quad f(X) = \max_{i \leq n} X_i.
\end{align*}
\]

Output: \text{A NUMBER}
Embretchts and Puccetti (2006) introduced a multivariate approach of the Value at Risk,
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- **Multivariate lower-orthant Value at Risk**

  \[
  \text{VaR}_\alpha(X) := \partial\{x \in \mathbb{R}^n \mid F_X(x) \geq \alpha\}.
  \]

- **Multivariate upper-orthant Value at Risk**

  \[
  \overline{\text{VaR}}_\alpha(X) := \partial\{x \in \mathbb{R}^n \mid \bar{F}_X(x) \leq 1 - \alpha\}.
  \]

**Output:** A SURFACE ON \( \mathbb{R}^n \)
Cousin and Di Bernardino (2013) introduced a multivariate risk measure related to the measure introduced by Embrechts and Puccetti (2006).
Cousin and Di Bernardino (2013) introduced a multivariate risk measure related to the measure introduced by Embrechts and Puccetti (2006).

- **Multivariate lower-orthant Value at Risk**

\[
\text{VaR}_\alpha (X) := \mathbb{E} [X | F_X(x) = \alpha].
\]

- **Multivariate upper-orthant Value at Risk**

\[
\text{VaR}_\alpha (X) := \mathbb{E} [X | \bar{F}_X(x) = 1 - \alpha].
\]

**Output:** A POINT IN \( \mathbb{R}^n \)
The lack of a total order in high dimensions.
Drawbacks in the multivariate setting

- The lack of a total order in high dimensions.
- The dependence among the variables.
The lack of a total order in high dimensions.
The dependence among the variables.
There are many interesting directions to analyze the data.
The lack of a total order in high dimensions.
The dependence among the variables.
There are many interesting directions to analyze the data.
The computation in high dimensions.
Introduce a directional multivariate value at risk
**OBJECTIVES**

1. Introduce a directional multivariate value at risk.
2. Consider the dependence among the variables.
3. Give the possibility of analyzing the losses considering the manager preferences.
4. Improve the interpretation of the risk measure.
OBJECTIVES

1. Consider the dependence among the variables.
2. Give the possibility of analyzing the losses considering the manager preferences.
3. Improve the interpretation of the risk measure.
4. Provide a non-parametric estimation to compute the risk measure in high dimensions.
5. Provide analytic expressions of the risk measure with copulas.
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7. CONCLUSIONS
Let \( X \) be a random vector satisfying "the regularity conditions", then the Value at Risk of \( X \) in direction \( u \) and confidence parameter \( \alpha \) is defined as

\[
VaR^u_{\alpha}(X) = \left( Q_X(\alpha, u) \cap \{ \lambda u + E[X] \} \right),
\]

where \( \lambda \in \mathbb{R} \) and \( 0 \leq \alpha \leq 1 \).

Output: A POINT IN \( \mathbb{R}^n \)
\[ Q_X(\alpha, u) \equiv \text{Directional Multivariate Quantile (Laniado et al. (2012))} \]

**Definition**

Given \( u \in \mathbb{R}^n, \ ||u|| = 1 \) and a random vector \( X \) with distribution probability \( P \), the \( \alpha \)-quantile curve in direction \( u \) is defined as:

\[
Q_X(\alpha, u) := \partial \{ x \in \mathbb{R}^n : P [ \mathcal{C}_x^u ] \leq \alpha \},
\]

where \( \partial \) mans the boundary and \( 0 \leq \alpha \leq 1 \)
Given $x$, $u \in \mathbb{R}^n$ and $||u|| = 1$, the orthant with vertex $x$ and direction $u$ is:

$$C^u_x = \{z \in \mathbb{R}^n | R_u(z - x) \geq 0\},$$

where $e = \frac{1}{\sqrt{n}} (1, ..., 1)'$ and $R_u$ is a matrix such that $R_u u = e$. 

$C^u_x \equiv$ Oriented Orthant.
Examples of Oriented Orthants

(A) Orthant in direction $\mathbf{u} = (0, 1)$  
(B) Orthant in direction $\mathbf{u} = -\mathbf{e}$

Examples of oriented orthants in $\mathbb{R}^2$
\[ \mathbf{u} \in \mathcal{U} = \left\{ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\} \]

(A) Bivariate Uniform  (B) Bivariate Exponential  (C) Bivariate Normal

CLASSICAL DIRECTIONS
\[ \mathbf{u} \in \mathcal{U} = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \]

(A) Bivariate Uniform  (B) Bivariate Exponential  (C) Bivariate Normal

CANONICAL DIRECTIONS
Directional MVaR

**Directional Multivariate Value at Risk (MVaR)**

(A) Bivariate Uniform  (B) Bivariate Normal  (C) Bivariate Exponential

\[ \text{VaR}^{-e}_{0.7}(X) \]
Directional Multivariate Value at Risk (MVaR)

(A) Bivariate Uniform  (B) Bivariate Normal  (C) Bivariate Exponential

$$\text{VaR}^e_{0.3}(X)$$
**MVaR Properties**

- **Non-Negative Loading**: If \( \lambda > 0 \),

\[
\mathbb{E}[X] \preceq_u \text{VaR}_\alpha^u(X),
\]

where the order is given by

**Preorder (Laniado et al. (2010))**

\( x \) is said to be less than \( y \) if:

\[
x \preceq_u y \equiv \mathcal{C}_x^u \supseteq \mathcal{C}_y^u \equiv R_u x \leq R_u y.
\]
**MVAR PROPERTIES**

- **Quasi-Odd Measure:** \( \text{VaR}_\alpha^u(-X) = -\text{VaR}_\alpha^u(X) \).

- **Positive Homogeneity and Translation Invariance:** Given \( c \in \mathbb{R}^+ \) and \( b \in \mathbb{R}^n \), then

  \[ \text{VaR}_\alpha^u(cX + b) = c \text{VaR}_\alpha^u(X) + b. \]
**Orthogonal Quasi-Invariance**: Let \( w \) and \( Q \) be a unit vector and a particular orthogonal matrix obtained by a QR decomposition such that \( Qu = w \). Then,

\[
\text{VaR}_\alpha^w(QX) = Q\text{VaR}_\alpha^u(X).
\]
Consistency: Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors such that $\mathbb{E}[\mathbf{Y}] = cu + \mathbb{E}[\mathbf{X}]$, for $c > 0$ and $\mathbf{X} \leq \mathbb{E}_u \mathbf{Y}$. Then:

$$\text{VaR}^u_\alpha(\mathbf{X}) \preceq_u \text{VaR}^u_\alpha(\mathbf{Y}),$$

where the stochastic order is defined by

**Stochastic Extremality Order (Laniado et al. (2012))**

Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors in $\mathbb{R}^n$,

$$\mathbf{X} \leq \mathbb{E}_u \mathbf{Y} \equiv \mathbb{P}[R_u(\mathbf{X} - \mathbf{z}) \geq 0] \leq \mathbb{P}[R_u(\mathbf{Y} - \mathbf{z}) \geq 0] \equiv \mathbb{P}_X[\mathbb{C}_u^u] \leq \mathbb{P}_Y[\mathbb{C}_z^u],$$

for all $\mathbf{z}$ in $\mathbb{R}^n$. 

**Directional MVaR Properties**
**MVar Properties**

- **Non-Excessive Loading:** For all $\alpha \in (0, 1)$ and $u \in \mathbb{B}(0)$,
  \[
  \text{VaR}^u_\alpha (X) \preceq_u R'_u \sup_{\omega \in \Omega} \{R_u X(\omega)\}.
  \]

- **Subadditivity in the Tail Region:** Let $X$ and $Y$ be random vectors, with the same mean $\mu$ and let $(R_u X, R_u Y)$ be a regularly varying random vector. Then,
  \[
  \text{VaR}^u_\alpha (X + Y) \preceq_u \text{VaR}^u_\alpha (X) + \text{VaR}^u_\alpha (Y).
  \]
Let $\mathbf{X}$ be a random vector and $\mathbf{u}$ a direction. Then for all $0 \leq \alpha \leq 1$,

$$\text{VaR}_\alpha^\mathbf{u} (\mathbf{X}) \preceq_\mathbf{u} \text{VaR}_{1-\alpha}^{-\mathbf{u}} (\mathbf{X}).$$
Then, analogously as Embrechts and Puccetti (2006) and Cousin and Di Bernardino (2013), we can define:

**Lower Multivariate VaR in the direction** $u$ **as**

$$\text{VaR}^u_\alpha(X),$$

**Upper Multivariate VaR in the direction** $u$ **as**

$$\text{VaR}^{-u}_{1-\alpha}(X).$$
Lower Multivariate VaR = $\text{VaR}_{0.3}^{e}(X)$ and
Upper Multivariate VaR = $\text{VaR}_{0.7}^{-e}(X)$
Lower Multivariate VaR = $\text{VaR}_{0.3} \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) (X)$ and

Upper Multivariate VaR = $\text{VaR}_{0.7} \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) (X)$
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Relation between the marginal VaR and the MVaR

Result

Let $\mathbf{X}$ be a random vector with survival function $\bar{F}$ quasi-concave. Then, for all $\alpha \in (0, 1)$:

$$VaR_{1-\alpha}(X_i) \geq [VaR^e_\alpha(\mathbf{X})]_i, \quad \text{for all} \quad i = 1, \ldots, n.$$ 

Moreover, if its distribution function $F$ is quasi-concave, then, for all $\alpha \in (0, 1)$,

$$[VaR_{1-\alpha}^{-e}(\mathbf{X})]_i \geq VaR_{1-\alpha}(X_i), \quad \text{for all} \quad i = 1, \ldots, n.$$
RESULT

Let $X$ be a random vector and $u$ a direction. If the survival function of $R_uX$ is quasi-concave. Then, for all $0 \leq \alpha \leq 1$,

$$VaR_{1-\alpha}([R_uX]_i) \geq [R_uVaR^u_{\alpha}(X)]_i,$$

for all $i = 1, \ldots, n$.

And if $R_uX$ has a quasi-concavity cumulative distribution, we have that

$$[R_uVaR^{-u}_{1-\alpha}(X)]_i \geq VaR_{1-\alpha}([R_uX]_i),$$

for all $i = 1, \ldots, n$. 

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Bivariate Copulas

- $\mathbb{R}^2 \Rightarrow u = (\cos \theta, \sin \theta)$.
- Let $X$ be a bivariate vector with density given by a copula density $c(\cdot, \cdot)$. Then, the first component of $\text{VaR}_\alpha^u(X)$ can be obtained by solving the equation on the domain,

$$\int \int_{D_\theta(x_1)} c(s, t) dt ds = \alpha,$$

where $D_\theta(x_1) = \mathcal{C}_{(x_1, l_\theta(x_1))} \cap [0, 1]^2$ and

$$l_\theta(x_1) := \begin{cases} \frac{x_1 \sin(\theta) - \frac{1}{2} (\sin(\theta) - \cos(\theta))}{\cos(\theta)}, & \text{if } \cos(\theta) \neq 0 \text{ and } x_1 \in [0, 1], \\ \frac{1}{2}, & \text{if } \cos(\theta) = 0 \text{ and } x_1 \in [0, 1]. \end{cases}$$
Example of $D_{\theta}(x_1)$ for $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
Bivariate Copulas

Results of $\text{VaR}^u_\alpha(X)$ with the Frank’s copula, for different values on the dependence parameter $\beta$:

a) Direction $\mathbf{u} = -\mathbf{e}$

b) Direction $\mathbf{u} = -\frac{3\sqrt{5}}{5} [\frac{1}{3}, \frac{2}{3}]'$

Behavior of the first component of $\text{VaR}^u_\alpha(X)$
Let $X$ be a $n$-dimensional random vector with $[0, 1]$-uniform marginals.

- If $X$ has an Archimedean copula distribution generated by $\phi(\cdot)$, then:

$$[\text{VaR}_{1-\alpha}^e(X)]_i = \phi^{-1} \left( \frac{1 - \phi(\alpha)}{n} \right).$$

- If $X$ has a survival copula given by an Archimedean copula generated by $\bar{\phi}(\cdot)$, then:

$$[\text{VaR}_\alpha(X)]_i = 1 - \bar{\phi}^{-1} \left( \frac{\bar{\phi}(\alpha)}{n} \right).$$
Then, we compare $V a R_{\alpha}^{-e}(X)$ (Our) with $V a R_{\alpha}(X)$ (Cousin and Di Bernardino (2013)) and $V a R_{1-\alpha}^{e}(1-X)$ with $V a R_{\alpha}(1-X)$, using the Clayton’s family of copulas.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{copulas.png}
\caption{Comparison of VaR with Clayton's family of copulas.}
\end{figure}

\begin{itemize}
\item[a)] Lower Case
\item[b)] Upper Case
\end{itemize}

Dashed line $\equiv$ Cousin and Di Bernardino. Solid line $\equiv$ $V a R_{\alpha}^{u}(X)$
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Given the sample $X_m := \{x_1, \cdots, x_m\}$ of the random loss $X$, the direction $u$ and the value of $\alpha$. We find the directional quantile curve as:

$$\hat{Q}_{X_m}(\alpha, u) := \{x_i : \mathbb{P}_{X_m}[c_{X_i}^u] = \alpha\},$$

where

$$\mathbb{P}_{X_m}[c_{X_i}^u] = \frac{1}{m} \sum_{j=1}^{m} \mathbb{1}\{x_j \in c_{X_i}^u\}.$$
However, it is possible that $\hat{Q}_{X_m}(\alpha, u) = \emptyset$. This can be solved allowing a slack $h$:

$$\hat{Q}_{X_m}^h(\alpha, u) := \left\{ x_j : \left| \mathbb{P}_{X_m} \left[ C_{X_j}^u \right] - \alpha \right| \leq h \right\},$$

where $\hat{Q}_{X_m}(\alpha, u) \subset \hat{Q}_{X_m}^h(\alpha, u)$, for all $h$.

Once the directional $\alpha$-quantile curve is obtained, we cross it with the line $\{ \mu_{X_m} + \lambda u \}$ where

$$\mu_{X_m} = \mathbb{E}[X_m].$$
Non-Parametric Estimation

**Non-Parametric Estimation**

Input: $u$, $\alpha$, $h$ and the multivariate sample $X_m$. 

for $i = 1$ to $m$

$P_i = \mathbb{P}_{X_m}[c_{x_i}^u]$, 

if $|P_i - \alpha| \leq h$

$x_i \in \hat{Q}_h^{X_m}(\alpha, u)$, 

end

for $x_j \in \hat{Q}_h^{X_m}(\alpha, u)$

$d_j = \text{dist}(x_j, \{\mu_{X_m} + \lambda u\})$, 

end

end

$\text{VaR}_u^{\alpha}(X_m) = \{x_k | d_k = \min\{d_j\}\}$. 

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Multivariate VaR: Directional perspective
## Execution Time

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</tbody>
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In an Intel core i7 (3.4 GH) computer with 32 Gb RAM.
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7 CONCLUSIONS
We analyze the behavior of the *MVaR* when a sample is contaminated with different types of outliers.

We use as a benchmark the measurement given by the multivariate *VaR* in Cousin and Di Bernardino (2013).
We simulate 5000 observations of the following random vector:

\[
X^\omega \overset{d}{=} \begin{cases} 
X_1 \quad \text{with probability } p = 1 - \omega, \\
X_2 \quad \text{with probability } p = \omega,
\end{cases}
\]

where \(X_1 \overset{d}{=} N_1(\mu_1, \Sigma_1), X_2 \overset{d}{=} N_2(\mu_1 + \Delta \mu, \Sigma_1 + \Delta \Sigma)\) and \(0 \leq \omega \leq 1\). Specifically:

\[
\mu_1 = [50, 50]', \quad \Sigma_1 = \begin{pmatrix} 0.5 & 0.3 \\
0.3 & 0.5 \end{pmatrix}.
\]

Contaminating \(\begin{cases} 
1. \text{Varying only the mean.} \\
2. \text{Varying only the variances.} \\
3. \text{Varying all the parameters.}
\end{cases}\)
To evaluate the impact of the contamination, we use:

\[ PV^\omega = \frac{||\text{Measure}(X^\omega) - \text{Measure}(X^0)||_2}{||\text{Measure}(X^0)||_2}, \]

where \( \text{Measure}(X^0) \) is the sample with \( \omega = 0\% \) and \( \text{Measure}(X^\omega) \) is the sample with level of contamination \( \omega\% \), \( (\omega = 1\% \rightarrow 10\%) \).
Robustness

1. Varying only the mean, $\Delta \mu \neq 0, \quad \Delta \Sigma = 0$.

(A) $\Delta \mu = (20, 20)'$

(B) $\Delta \mu = (0, 50)'$

Mean of $PV^\omega$
2. Varying only the variances, $\Delta \mu = 0$, 

$$\Delta \Sigma = \begin{bmatrix} 4.5 & 0 \\ 0 & 6.5 \end{bmatrix},$$

Mean of $PV^\omega$
3. Varying all the parameters, \( \Delta \mu \neq 0, \quad \Delta \Sigma = \begin{bmatrix} 4.5 & 0.2 \\ 0.3 & 6.5 \end{bmatrix} \),

\[
\begin{align*}
(A) \quad & \Delta \mu = (20, 20)'
\end{align*}
\]

\[
\begin{align*}
(B) \quad & \Delta \mu = (0, 50)'
\end{align*}
\]

Mean of \( PV^\omega \)
Conclusions

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Conclusions

We introduce a directional multivariate value at risk and a non-parametric estimation for this risk measure.
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The directional approach allows to consider external information or management preferences in the analysis of the data.
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We provide good properties for this risk measure, including the tail subadditivity property.
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We obtain analytic expressions with copulas.
Conclusions

We introduce a directional multivariate value at risk and a non-parametric estimation for this risk measure.

The directional approach allows to consider external information or management preferences in the analysis of the data.

We provide good properties for this risk measure, including the tail subadditivity property.

We obtain analytic expressions with copulas.

The simulation study of robustness shows good behavior of the measure.
Thanks
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