Riemannian Wave-field Extrapolation
Thesis Proposal

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Outline

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2. Wave propagation in Continuum media
3. The Problem
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5. Objectives

Introduction

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Introduction

- *The earth is at least a visco elastic medium, in which absorption losses give rise to attenuation and dispersion effects.*
- *The elastic wave equation is framed in terms of tensor operators acting on vector quantities.*
- *...it is also true that a proper treatment of anisotropy fundamentally demands an elastic viewpoint, even when only P-waves (quasi-P waves) are contemplated.*
- *....different representations for the same physical law can lead to different computational techniques in solving the same problem, which can produce different and new numerical results, so this new but accurate representation should lead us to new results and descriptions of the phenomena.*

Wave Propagation in Continuum Media

- Hook's law
- Cauchy's equations of motion
- Wave equation for P-waves in homogeneous and isotropic media
- Wave equation for S-waves in homogeneous and isotropic media

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Wave Propagation in Continuum Media

- Hook's law

$$\sigma_{ij} = \sum_{k,l} \mathcal{C}_{ijkl} \; \epsilon_{kl}$$

where

$$\begin{aligned} \sigma_{ij} &: \quad \text{is the strain tensor,} \\ \mathcal{C}_{ijkl} &: \quad \text{is the stiffnes tensor,} \\ \epsilon_{kl} &: \quad \text{is the stress tensor.} \end{aligned}$$

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  From the balance of momentum one gets

$$\rho(\vec{x}) \frac{\partial^2 \vec{u}_i}{\partial t^2} = \sum_j \frac{\partial}{\partial x_j} \sigma_{ij}$$

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  For an Isotropic media

$$\sigma_{ij} = \lambda\delta_{ij}\sum_k \epsilon_{kk} + 2\mu\epsilon_{ij}$$

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$$\rho(\vec{x})\frac{\partial^2 \vec{u}_i}{\partial t^2} = \sum_j \frac{\partial}{\partial x_j}\sigma_{ij}$$

then

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)[\triangledown(\triangledown \cdot \vec{u})] + \mu \triangledown^2 \vec{u}$$

- Wave equation for P-waves in homogeneous and isotropic media
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$$\rho(\vec{x})\frac{\partial^2 \vec{u}_i}{\partial t^2} = \sum_j \frac{\partial}{\partial x_j}\sigma_{ij}$$

In general curvilinear coordinates

$$\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u})$$

and defining

$$\begin{aligned} \varphi &= \nabla \cdot \vec{u} \\ \psi &= \nabla \times \vec{u} \end{aligned}$$

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  we get

$$\rho(\vec{x})\frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + 2\mu) \bigtriangledown \varphi - \mu \bigtriangledown \times \psi$$

- Wave equation for P-waves in homogeneous and isotropic media
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Wave Propagation in Continuum Media

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Wave Propagation in Continuum Media

- Hook's law
- Cauchy's equations of motion
- Wave equation for P-waves in homogeneous and isotropic media

$$\nabla^2\varphi - \frac{1}{v_p^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

where

$$v_p = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}}$$

- Wave equation for S-waves in homogeneous and isotropic media

Wave Propagation in Continuum Media

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Wave Propagation in Continuum Media

- Hook's law
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- Wave equation for S-waves in homogeneous and isotropic media

$$\bigtriangledown^2\psi - \frac{1}{v_s^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

where

$$v_s = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$$

On Wave equation

On Wave equation

Consider the IVP

$$\begin{aligned} \bigtriangledown^2 \vec{u} - \frac{1}{v^2}\frac{\partial^2 \vec{u}}{\partial t^2} &= 0 \\ \vec{u}(\vec{x},0) &= \gamma(\vec{x}) \\ \frac{\partial \vec{u}}{\partial t}|_{t=0} &= \eta(\vec{x}) \end{aligned}$$

On Wave equation

- In one dimension (1-D)

On Wave equation

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$$u(x,t) = \frac{1}{2}\left[\gamma(x+vt) + \gamma(x-vt) + \frac{1}{v}\int_{x-vt}^{x+vt} \eta(s)ds\right]$$

where

$$\begin{aligned} \gamma(x) &= f(x) + g(x) \\ \eta(x) &= v[f'(x) + g'(x)] \end{aligned}$$

for some $f, g \in \mathcal{C}^2(\Omega)$

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- In one dimension (1-D)
- In two dimensions (2-D)

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- In two dimensions (2-D)

$$
\begin{aligned}
\vec{u}(\vec{x}, t) &= \frac{d}{dt}\left[\frac{4\pi^2}{v}\iint_{D(\vec{x},vt)} \frac{\gamma(s_1, s_2)}{\sqrt{(vt)^2 - [(s_1 - x_1)^2 + (s_2 - x_2)^2]}} ds_1 ds_2\right] \\
&+ \frac{4\pi^2}{v}\iint_{D(\vec{x},vt)} \frac{\eta(s_1, s_2)}{\sqrt{(vt)^2 - [(s_1 - x_1)^2 + (s_2 - x_2)^2]}} ds_1 ds_2
\end{aligned}
$$

Elasticity Theory

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Elasticity Theory

- A configuration on $\mathcal{B}$ is a smooth, orientation preserving and invertible mapping

$$\Phi : \mathcal{B} \to \int$$

  The set of all configurations of $\mathcal{B}$ is denoted $\mathcal{C}$
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- We denote motions as $\Phi(X, t)$, where $X \in \mathcal{B}$ and $x = \Phi(X) \in \mathcal{S}$
- The material velocity and acelerations are defined as (for X fixed)

$$\begin{aligned} V_t(X) &= \frac{\partial}{\partial t}\Phi(X, t) \\ A_t(X) &= \frac{\partial}{\partial t}V_t(X) \end{aligned}$$

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$$\begin{aligned} V_t(X) &= \frac{\partial}{\partial t}\Phi(X,t) \\ A_t(X) &= \frac{\partial}{\partial t}V_t(X) \end{aligned}$$

- The spatial velocity and acelerations are defined as (for t fixed)

$$\begin{aligned} v_t &:= V_t \circ \mathit{\Phi}^{-1} \\ a_t &:= A_t \circ \mathit{\Phi}^{-1} \end{aligned}$$

Elasticity Theory

Elasticity Theory

- The deformation gradient, is given by

$$\begin{aligned} F : T\mathcal{B} &\rightarrow T\mathcal{S} \\ F(X, W) &= (\Phi(X), D\Phi(x) \cdot W) \end{aligned}$$

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Elasticity Theory

-
- The right Cauchy-Green tensor is given by

$$
\begin{aligned}
C : T_X\mathcal{B} &\rightarrow T_X\mathcal{B} \\
C(X,W) &= \left(X, D\Phi(X)^T D\Phi(X)\cdot W\right) \\
C(X) &= F^T(X)F(X)
\end{aligned}
$$

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- some properties of C
  1. C is Symmetric
  2. C is semi-positive definite
  3. If every F is one-to one, then C is positive definite and invertible.
-
-

Elasticity Theory

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- The left Cauchy-Green tensor is given by

$$\begin{aligned} b: T_x\Phi(\mathcal{B}) &\rightarrow T_x\Phi(\mathcal{B}) \\ b(x) &= F(X)F^T(X) \end{aligned}$$

-

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$$

- some properties of b
  1. b is Symmetric
  2. b is positive definite

Elasticity Theory

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Elasticity Theory

- Consider the symmetric, positive definite, linear transformations U, V such that

$$
\begin{aligned}
U^2 &= C \\
V^2 &= b
\end{aligned}
$$

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- It can be shown that (polar decomposition of F)

$$F = RU = VR$$

for some unique orthogonal transform

$$R : T_X\mathcal{B} \to T_x\mathcal{S}$$

and

$$U = R^T VR$$

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Elasticity Theory

- Consider the symmetric, positive definite, linear transformations U, V such that

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$$R : T_X\mathcal{B} \to T_x\mathcal{S}$$

and

$$U = R^T VR$$

- The Strain tensor is given by

$$\begin{aligned} E : T\mathcal{B} &\to T\mathcal{B} \\ E &= \frac{1}{2}\left[C - Id\right] \end{aligned}$$

To propose a Riemannian wavefield propagation theory which accounts for general symmetries of the medium and to propose decoupled solutions of the general Riemannian wavefield equation which can be applied in migration algorithms, in particular to one way wave equation (OWWE) algorithms. The existing theory has not reseached a point in which they can describe general continuum, complex zones, and used these theorical descriptions in migration algorithms, the theory needed is a mixture of differential geometry, functional analysis and migration methods.

OWWE. Extrapolation methods

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- Phase-shift (J.Gazdag)

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$$
\begin{aligned}
\varphi(k_x, z_j, \omega) &= \varphi(k_x, z_{j-1}, \omega) e^{ik_z \Delta z} \\
\varphi(k_x, z, \omega) &= \mathcal{F}[\psi(x, z, \omega)] \\
\varphi(k_x, z_0, \omega) &:= \text{Data}
\end{aligned}
$$

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$$
\begin{aligned}
s(\vec{r}, z) &= \frac{2}{v(\vec{r}, z)} \\
\bigtriangledown^2 \varphi + \omega^2 s^2 &= 0 \\
s(\vec{r}, z) &= s_0(z) + \bigtriangleup s(\vec{r}, z) \\
\bigtriangledown^2 \varphi + \omega^2 s_0^2(z) \varphi &= -S(\vec{r}, z, \omega)
\end{aligned}
$$

OWWE. Extrapolation methods

- Phase-shift (J.Gazdag)
- Split-Step Fourier Migration (P.L. Stoffa)

$$\begin{aligned}
\frac{\partial^2}{\partial z^2} P(k_r, z, \omega) + K_{z_0}^2 P(k_r, z, \omega) &= -\hat{S}(k_r, z, \omega) \\
P_-(\vec{r}, z_{n+1}, \omega) &= P_l(\vec{r}, z_n, \triangle\, z, \omega) \\
&+ i\omega \int_{z_n}^{z_{n+1}} \triangle\, s P_l(\vec{r}, z', d_{n+1}, \omega) dz'
\end{aligned}$$

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$$
\begin{aligned}
\left[\frac{\partial}{\partial z} + i\sqrt{A(x,\omega)}\right]\left[\frac{\partial}{\partial z} - i\sqrt{A(x,\omega)}\right]\varphi(x,z,\omega) &= 0 \\
A(x,\omega) &= \frac{\partial^2}{\partial x^2} + \frac{\omega^2}{v^2(x,z_j)} \\
s(x,z_j) &= \frac{1}{v(x,z_j)}
\end{aligned}
$$

with the extrapolators

$$
\begin{aligned}
k_z &= \sqrt{\omega^2 s^2 - k_x^2} \\
k_{z0} &= \sqrt{\omega^2 s_0^2 - k_x^2}
\end{aligned}
$$

we get

$$
k_z = k_{z0}\sqrt{1 - \frac{\omega^2}{k_{z0}^2}(s_0^2 - s^2)} \quad (1)
$$

OWWE. Extrapolation methods

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$$k_z = k_{z_0} + k_{z_0} \sum_{n=1}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} \left[\left(\frac{\omega^2 s_0^2}{\omega^2 s_0^2 - k_x^2} \right) \left(\frac{s_0^2 - s^2}{s_0^2} \right) \right]^n$$

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$$\psi(x, z + \triangle z, \omega) = \psi(x, z, \omega) e^{ik_{z_0} \triangle z} e^{ik_{z_0} \triangle z} \sum_{n=1}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} \left[\left(\frac{\omega^2 s_0^2}{\omega^2 s_0^2 - k_x^2} \right) \left(\frac{s_0^2 - s^2}{s_0^2} \right) \right.$$

$$\psi(x, z + \triangle z, \omega) = \psi(x, z, \omega) e^{ik_{z_0} \triangle z} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} \left[\left(\frac{\omega^2 s_0^2}{\omega^2 s_0^2 - k_x^2} \right) \left(\frac{s_0^2 - s^2}{s_0^2} \right) \right] \right.$$

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  For the self-adjoint operator

$$\mathcal{L} = -\left(\frac{2\pi\omega}{v(x,z)}\right)^2 - D_{xx} - D_{yy}$$

Construct the spectral family (spectral projectors)

$$\begin{aligned} \mathcal{P} &= \sum_{(k:\lambda_k \leq 0)} \lambda_k P_k \\ \mathcal{P}\mathcal{L}\mathcal{P} &= \sum_{(k:\lambda_k \leq 0)} \lambda_k P_k \end{aligned}$$

OWWE. Extrapolation methods

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- Full-Wave-Equation depth extrapolation (K.Sandberg, G.Beylkin) reformulate the problem as

$$
\begin{aligned}
\hat{p}_{zz} &= \mathcal{P}\mathcal{L}\mathcal{P}\hat{p} \\
\hat{p}(x, z_n, \omega) &= q(x, z_n, \omega) \\
\hat{p}_z(x, z_n, \omega) &= q_z(x, z_n, \omega)
\end{aligned}
$$

Riemannian wavefield extrapolation, finite difference approach

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- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge)

Riemannian wavefield extrapolation, finite difference approach

- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge)
  Consider the monochromatic wave equation for an acoustic wavefield

$$\begin{aligned} \nabla_\xi^2 \mathcal{U} &= -\omega^2 s_\xi^2 \mathcal{U}, \text{where} \\ \nabla_\xi^2 \mathcal{U} &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \xi_i} \left(\sqrt{|g|} g^{ij} \frac{\partial \mathcal{U}}{\partial \xi_j} \right) \end{aligned}$$

Riemannian wavefield extrapolation, finite difference approach

- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge)
  This equation can be written as

$$n^j \frac{\partial \mathcal{U}}{\partial \xi_j} + m^{ij} \frac{\partial^2 \mathcal{U}}{\partial \xi_i \partial \xi_j} = -\sqrt{|g|}\omega^2 s_\xi^2 \mathcal{U}$$

  where

$$n^j \ , \ m^{ij} \quad \text{depend on the metric.}$$

- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge)
  Fourier transforming $\xi_\nu \leftrightarrow k_\nu$

  $$(m^{ij}k_{\xi_i} - in^j)k_{\xi_j} = \sqrt{|g|}\omega^2 s_\xi^2,$$

  Solving for k_{ξ_3} leads to

  $$k_{\xi_3} = -a_1 k_{\xi_1} - a_2 k_{\xi_2} + ia_3 \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} + ia_8 k_{\xi_1} + ia_9 k_{\xi_2} - a_{10}^2]^{1/2}$$

  and then extrapolate

  $$\mathcal{U}(\xi_3 + \triangle\,\xi_3, k_{\xi_1}, k_{\xi_2}, \omega) = \mathcal{U}(\xi_3, k_{\xi_1}, k_{\xi_2}, \omega)e^{ik_{\xi_3}\triangle\xi_3}$$

  Some extrapolators

Riemannian wavefield extrapolation, finite difference approach

- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge) 2D nonorthogonal coordinate system.

$$k_{\xi_3} = -a_1 k_{\xi_1} + ia_3 \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 + ia_8 k_{\xi_1} - a_{10}^2]^{1/2}$$

Riemannian wavefield extrapolation, finite difference approach

- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge) 2D orthogonal coordinate system.

$$k_{\xi_3} = ia_3 \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 + ia_8 k_{\xi_1} - a_{10}^2]^{1/2}$$

Riemannian wavefield extrapolation, finite difference approach

- Riemannian wavefield extrapolation (P.Sava, S.Fomel, J.Shragge)
  3D semiorthogonal coordinate system.

$$k_{\xi_3} = ia_3 \pm [a_4^2\omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} + ia_8 k_{\xi_1} + ia_9 k_{\xi_2} - a_{10}^2]^{1/2}$$

Finite Difference Scheme for the Riemannanian 2D acoustic wave equation

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$$
\begin{aligned}
\left[\nabla_\xi^2 - \frac{1}{\nu_\xi^2}\frac{\partial^2}{\partial t^2}\right] U_\xi &= F_\xi \\
\nabla_\xi^2 &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial \xi_i}\left(g^{ij}\sqrt{|g|}\right)\frac{\partial}{\partial \xi_j} + g^{ij}\frac{\partial^2}{\partial \xi_i \partial \xi_j} \\
\nabla_\xi^2 &= \zeta^i \frac{\partial}{\partial \xi_i} + g^{ij}\frac{\partial^2}{\partial \xi_i \partial \xi_j}
\end{aligned}
$$

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  Then, we have

$$\zeta^i \frac{\partial U_\xi}{\partial \xi_i} + g^{ij} \frac{\partial^2 U_\xi}{\partial \xi_i \partial \xi_j} = \frac{1}{\nu_\xi^2} \frac{\partial^2 U_\xi}{\partial t^2} + F_\xi$$

Finite Difference Scheme for the Riemannanian 2D acoustic wave equation

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  For a 2D scheme, we have

$$\frac{\partial^2 U_\xi}{\partial t^2} = \nu^2 \left[\zeta^1 \frac{\partial U_\xi}{\partial \xi_1} + \zeta^2 \frac{\partial U_\xi}{\partial \xi_2} + g^{11} \frac{\partial^2 U_\xi}{\partial \xi_1^2} + 2g^{12} \frac{\partial^2 U_\xi}{\partial \xi_1 \partial \xi_2} + g^{22} \frac{\partial^2 U_\xi}{\partial \xi^2} \right]$$

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  Take the following FD scheme

$$
\begin{aligned}
\frac{\partial^2 U}{\partial t^2} &= \frac{U_{v,k}^{n+1} - 2U_{v,k}^{n} + U_{v,k}^{n-1}}{(\Delta t)^2} \\
\frac{\partial U}{\partial \xi_1} &= \frac{U_{v+1,k}^{n} - U_{v-1,k}^{n}}{2\Delta \xi_1} \\
\frac{\partial U}{\partial \xi_1 \partial \xi_2} &= \frac{U_{v+1,k+1}^{n} - U_{v-1,k+1}^{n} - U_{v+1,k-1}^{n} + U_{v-1,k-1}^{n}}{2\Delta \xi_1 \Delta \xi_2} \\
\frac{\partial^2 U}{\partial \xi_1^2} &= \frac{U_{v+1,k}^{n} - 2U_{v,k}^{n} + U_{v-1,k}^{n}}{(\Delta \xi_1^2)} \\
\frac{\partial^2 U}{\partial \xi_2^2} &= \frac{U_{v,k+1}^{n} - 2U_{v,k}^{n} + U_{v,k-1}^{n}}{(\Delta \xi_2^2)}
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_1 &= v\Delta \xi_1 \\
\xi_2 &= k\Delta \xi_2 \\
t &= n\Delta t \\
U_{v,k}^{n} &= U(\xi_1, \xi_2, t)
\end{aligned}
$$

- Finite Difference Scheme for the Riemannanian 2D acoustic wave equation
  So we obtain the following discrete equation

$$
\begin{aligned}
U^n_{v,k} &= 2U^n_{v,k} - U^{n-1}_{v,k} + (\nu\Delta t)^2[\zeta^1\left(\frac{U^n_{v+1,k} - U^n_{v-1,k}}{2\Delta\xi_1}\right) \\
&+ \zeta^2\left(\frac{U^n_{v,k+1} - U^n_{v,k-1}}{2\Delta\xi_2}\right) \\
&+ g^{11}\left(\frac{U^n_{v+1,k} - 2U^n_{v,k} + U^n_{v-1,k}}{(\Delta\xi_1)^2}\right) \\
&+ g^{22}\left(\frac{U^n_{v,k+1} - 2U^n_{v,k} + U^n_{v,k-1}}{(\Delta\xi_2)^2}\right) \\
&+ g^{12}\left(\frac{U^n_{v+1,k+1} - U^n_{v-1,k+1} + U^n_{v+1,k-1} + U^n_{v-1,k-1}}{2\Delta\xi_1\Delta\xi_2}\right)]
\end{aligned}
$$

On Elastic Wave equation

On Elastic Wave equation

If we want to have an elastic two-way equation

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On Elastic Wave equation

$$T_{ik}u_{k,33} - i\omega(R_{ik} + R_{ki}) - \omega^2 Q_{ik}u_k + \rho\omega^2 u_i = 0,$$

where

-
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-

On Elastic Wave equation

$$
\begin{aligned}
T_{ik} &= C_{i3k3} \\
R_{ik} &= C_{i1k3}s_1 + C_{i2k3}s_2 \\
Q_{ik} &= C_{i1k1}s_1^2 + (C_{i1k2} + C_{i2k1})s_1 s_2 + C_{i2k2}s_2^2.
\end{aligned}
$$

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On Elastic Wave equation

Note that in matrix notation, we have

$$T\frac{d^2\vec{u}}{dz^2} - i\omega(R + R^T)\frac{d\vec{u}}{dz} - \omega^2(Q - \rho I)\vec{u} = 0.$$

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-

On Elastic Wave equation

- Gluing together the momentum and constitutive equations, we have

$$\frac{d\vec{b}}{dz} = i\omega A\vec{b}$$

  where
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-

On Elastic Wave equation

- $$\vec{b} = \begin{pmatrix} \vec{u} \\ \vec{\tau} \end{pmatrix}; \quad \vec{\tau} = -\frac{1}{i\omega} \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix},$$

  and
-
-
-

On Elastic Wave equation

- $$A = -\begin{pmatrix} T^{-1}R^T & T^{-1} \\ RT^{-1}R^T - Q + \rho I & RT^{-1} \end{pmatrix}.$$
-
-
-

On Elastic Wave equation

-
- Matrix A can be decomposed as

$$D^{-1}AD = \Lambda = \mathrm{diag}(q_p^U \quad q_{s1}^U \quad q_{s2}^U \quad q_p^D \quad q_{s1}^D \quad q_{s2}^D),$$

-
-

On Elastic Wave equation

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- For a vertically homogeneous layer, we have $\vec{b} = D\vec{v}$ and the system reduces to

$$\frac{d\vec{v}}{dz} = i\omega\Lambda\vec{v},$$

  whose solution has the form
-

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-
- $$\vec{v}(z) = e^{i\omega\Lambda(z-z_0)}\vec{v}(z_0).$$

  Since $\vec{v} = D^{-1}\vec{b}$, which means that D^{-1} is a decomposition operator, we have

  $$\vec{b}_i(z) = D_i e^{i\omega\Lambda_i(z-z_0)} D_i^{-1}\vec{b}_i(z_0), \qquad (1)$$

An example

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An example

- For small motions of $\mathcal{B}$, we have

$$\Phi_t^i(X) = x^i + u^i(X,t)$$

  where $u = \sum u^i(X,t)\partial_i$ is the displacement vector field.
-
-
-

An example

-
- The strain tensor ε_{ij} is given by

$$\varepsilon_{ij} dx^i \otimes dx^j = \frac{1}{2}\left[*ds(X)^2 - ds(X)^2 \right]$$

, then

$$\varepsilon_{kl} = \frac{1}{2}\left(g_{km}\partial_l u^m + g_{ml}\partial_k u^m + u^m \partial_m g_{kl} \right)$$

$$\varepsilon_{kl} = \frac{1}{2}\left(g_{km} \bigtriangledown_l u^m + g_{ml} \bigtriangledown_k u^m \right)$$

-
-

An example

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- Since

$$
\begin{aligned}
\frac{\sigma_{ij}}{\sqrt{|g|}} &= C_{ijkl}\varepsilon_{kl} \\
df^{i} &= \frac{\sigma_{ij}}{\sqrt{|g|}} dS_j
\end{aligned}
$$

we have, for an elastic and homogeneous body, the equation of motion given by:

$$
\begin{aligned}
\int_V \rho \partial_{tt} u^i dV &= -\int_S \frac{\sigma_{ij}}{\sqrt{|g|}} dS_j \\
&= \int_V \nabla_j \left(\frac{\sigma_{ij}}{\sqrt{|g|}} \right) dV.
\end{aligned}
$$

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An example

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- Then we have a elastic wave equation as

$$\begin{aligned}\rho\partial_{tt}u^i &= \frac{1}{2}\nabla_j C_{ijkl}\left(g_{km}\nabla_l u^m + g_{ml}\nabla_k u^m\right)\\ &= \frac{1}{2}C_{ijkl}\left(g_{km}\nabla_j\nabla_l u^m + g_{ml}\nabla_j\nabla_k u^m\right)\end{aligned}$$

Objectives

Objectives

To obtain an elastic Riemannian wave equation theory and its solutions, or approximate solutions, which describe the elastic wave propagation in general medium, taking into account the anisotropy parameters, the symmetry of the medium, yielding to a decoupling that can be applied in migration algorithms.

Objectives

- To design Riemannian coordinate systems that conform with the Euclidean ones in which a wavefield is to be extrapolated and propagate an acoustic Riemannian wavefield.

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- To formulate the theory of elastic wave propagation in Riemannian manifolds which include the anisotropy parameters and the simmetries of the media.

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- To formulate the theory of elastic wave propagation in Riemannian manifolds which include the anisotropy parameters and the simmetries of the media.
- To obtain the decoupling of the solutions to the Riemannian wave equation in terms of pseudodifferential operators and/or Fourier integral operators.

Objectives

- To design Riemannian coordinate systems that conform with the Euclidean ones in which a wavefield is to be extrapolated and propagate an acoustic Riemannian wavefield.
- To formulate the theory of elastic wave propagation in Riemannian manifolds which include the anisotropy parameters and the simmetries of the media.
- To obtain the decoupling of the solutions to the Riemannian wave equation in terms of pseudodifferential operators and/or Fourier integral operators.
- To show that the decoupling operators can be reduced to the propagation operators used in one way wave equation extrapolation such as GPSPI, NSPS and GPS.

Some references

1. Sava.P, Fomel.S. 2005.*Riemannian Wave Field Extrapolation*. Geophysics, 70,T45-T56.
2. Shragge. J. 2008. *Riemannian Wave Field Extrapolation: Nonorthogonal Coordinate Systems*: Geophysics, 73, T11-T21.
3. Yasutomi Y.2007.*Modified Elastic Wave Equations On Riemannian and Kahler Manifolds*. Pulb. RIMS, 43,471-504.
4. Shragge. J.2014. *Solving the 3D acoustic wave equation on generalized structured meshes: A finite-difference time-domain approach*. Geophysics, 79, 1-16.
5. Bale. R. 2006.*Elastic wave equation depth migration of seismic data for Isotropic and Azimuthally Anisotropic media*. Ph.D Thesis, University of Calgary.
6. Marsden. J, Hughes. T. 1983 *Mathematical Foundations of Elasticity.* DOVER PUBLICATIONS, INC, New York.