A naturally conservative formulation to numerically solve the two-dimensional Hyperbolic Conservation and Balance laws on triangular grids.
State of the art

Jorge Agudelo
In colaboration with:
PhD. Eduardo Abreu, PhD. John Pérez and PhD. José Sánchez

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Outline

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1. Hyperbolic problems
2. Finite Volume Method
3. One-dimensional Lagrangian-Eulerian Scheme (LE1D) for HCL and BL
4. One dimensional numerical experiments
5. A 2D extension on cartesian meshes of the LE1D scheme for HCL via BL

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6. Two dimensional numerical experiments
7. Work's scope

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$g(u) \equiv 0 \Rightarrow$ **Conservation Law:** $\frac{\partial u(\mathbf{x},t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial f(u(\mathbf{x},t))}{\partial x_i} = 0.$

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Integral form:

$$\frac{d}{dt} \int_\Omega u(\mathbf{x}, t) d\mathbf{x} + \int_{\partial\Omega} f(u) \cdot n ds = 0.$$

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- Divide the spatial domain into cells called "**finite volumes**" or "**grid cells**".
- Approximate the average values of the unknown function over each finite volume.
- The key: correctly approximate the flux function by means of an approximation function called **numerical flux**.

LE1D Scheme [**Abreu et al. 2018**]

Integral Tube [Douglas et al. 2000; Abreu et al. 2018]

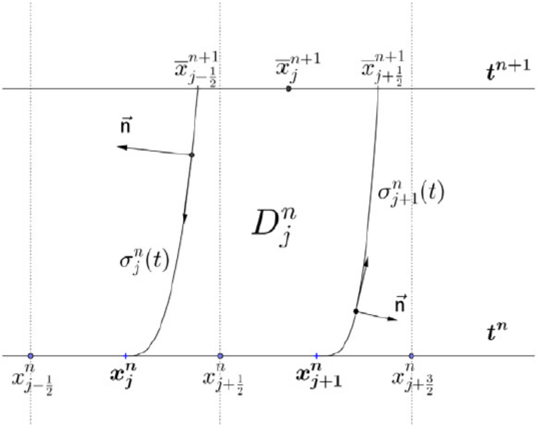

Figura 1: Integral Tube

LE1D Scheme [**Abreu et al. 2018**]

One-dimensional Conservation Law:

$$\begin{cases} u_t + f(u)_x = 0, & x \in [a,b],\ t > 0, \\ u(x,0) = u_0(x), & x \in [a,b]. \end{cases}$$

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Divergence form:

$$\nabla_{t,x} \cdot \begin{pmatrix} u \\ f(u) \end{pmatrix} = 0, \ \nabla_{t,x} = \begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

LE1D Scheme [Abreu et al. 2018]

Impermeability condition:

$$\int_{\sigma_j^n} \begin{pmatrix} u \\ f(u) \end{pmatrix} \cdot \overrightarrow{n} \, d\sigma_j^n = 0.$$

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IVP for Non-flow curves:

$$\begin{cases} \dfrac{d\sigma_j^n(t)}{dt} = \dfrac{f(u)}{u} & t^n < t \leq t^{n+1}, \\ \sigma_j^n(t^n) = x_j^n, & \end{cases} \qquad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}.$$

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A simple approximation of non-flow curves:

$$\sigma_j^n(t) = \frac{f(U_j^n)}{U_j^n}(t - t^n) + x_j^n, \;\; \forall n \in \mathbb{N}, \; \forall j \in \mathbb{Z}.$$

LE1D Scheme [**Abreu et al. 2018**]

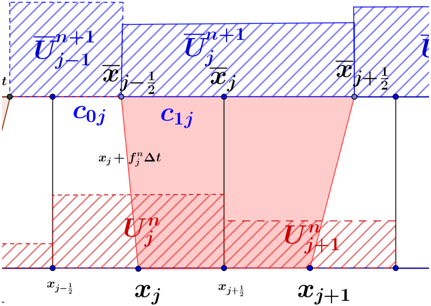

Figura 2: Approximate Integral Tube

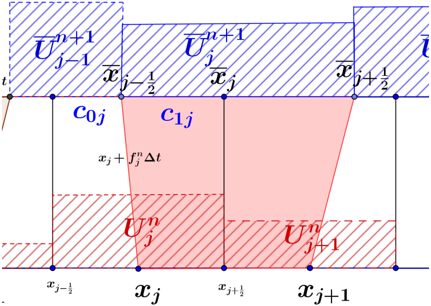

Figura 2: Approximate Integral Tube

$$\int_{D_j^n} \nabla_{t,x} \cdot \begin{pmatrix} u \\ f(u) \end{pmatrix} dA = 0 + \text{Divergence theorem} + \text{Impermeability}$$
condition:

LE1D Scheme [**Abreu et al. 2018**]

Local conservation law:

$$\int_{\overline{x}^{n+1}_{j-\frac{1}{2}}}^{\overline{x}^{n+1}_{j+\frac{1}{2}}} u(x, t^{n+1})dx = \int_{x^n_j}^{x^n_{j+1}} u(x, t^n)dx.$$

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Local conservation law + Approximate integral tube:

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LE1D scheme

$$U_0 = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} u_0(x)dx,$$

$$\begin{cases} \textbf{Evolution:} & \overline{U}_i^{n+1} = \frac{h}{h_i^{n+1}}\left(\frac{1}{2}U_i^n + \frac{1}{2}U_{i+1}^n\right), \\ \textbf{Projection:} & U_i^{n+1} = \frac{1}{h}[(\frac{h}{2} + f_i^n\Delta t)\overline{U}_{i-1}^{n+1} + \left(\frac{h}{2} - f_i^n\Delta t\right)\overline{U}_i^{n+1}], \end{cases}$$

$f_i^n = \frac{f(U_i^n)}{U_i^n}$ and $h_i^{n+1} = h + (f_{i+1}^n - f_i^n)\Delta t.$

One-dimensional hyperbolic balance law:

$$u_t + f(u)_x = g(u), \;\; x \in [a, b], \; t \geq 0.$$

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Well-balanced property

A numerical scheme is said to be well-balanced if it fully satisfies a discrete version of (1).

The LE1D scheme for hyperbolic balance laws

LE1D scheme for HBL

$$\begin{cases} \textbf{Evolution:} & \overline{U}_i^{n+1} = \frac{1}{h_i^{n+1}}\left[\int_{x_i^n}^{x_{i+1}^n} u(x,t^n)dx + \iint_{D_i^n} g(u)dA \right], \\ \textbf{Projection:} & U_i^{n+1} = \frac{1}{h}[(\frac{h}{2} + f_i^n \Delta t)\overline{U}_{i-1}^{n+1} + \left(\frac{h}{2} - f_i^n \Delta t\right)\overline{U}_i^{n+1}], \end{cases}$$

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$$\iint_{D_i^n} g(u)dA \approx \Delta t g\left(U_i^n + \frac{\Delta t}{2}(g(U_i^n) - f(U_i^n)_x)\right)\left(\frac{\Delta t}{2}(f_{i+1}^n - f_i^n) + h\right).$$

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- Burgers equation with Greenberg–LeRoux's and Riccati's source terms.

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = \cos^2\left(\frac{\pi x}{2}\right), -1 < x < 1, \\ u_t + \left(\frac{u^2}{2}\right)_x = \pm 2\left(1 + \sin\left(\frac{\pi x}{10}\right)\right), -0{,}1 < x < 49{,}9. \end{cases}$$

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- The shallow-water system.

$$h_t + (hu)_x = 0, \quad (hu)_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghZ_x.$$

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- Broadwell's rarefied gas dynamics.

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ m \\ z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} m \\ z \\ m \end{pmatrix} = \frac{1}{\epsilon}\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\left(\rho^2 + m^2 - 2\rho z\right). \end{pmatrix}$$

First attempt

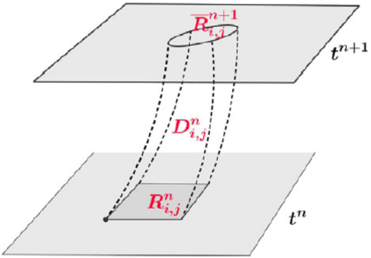

Figura 3: 2D Integral Tube

A 2D extension of the LE1D scheme for HCL via BL

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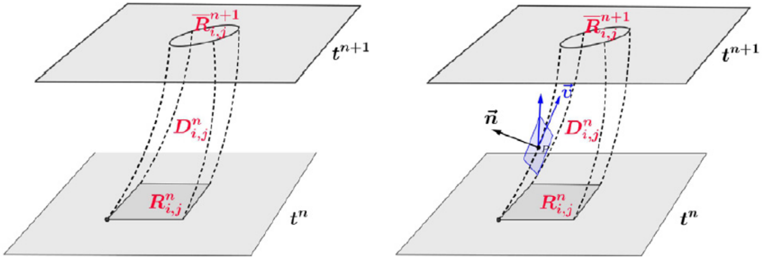

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Drawback: A new theory and feasibles numerical algorithms are needed.

An alternative approach

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Finite volume cells:

$$D_{i,j}^{n} = \left\{ (t,x,y)/t^{n} \leq t \leq t^{n+\frac{1}{2}}, y_{j-\frac{1}{2}}^{n} \leq y \leq y_{j+\frac{1}{2}}^{n}, \sigma_{i}^{n}(t) \leq x \leq \sigma_{i+1}^{n}(t) \right\},$$

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Coupled Set of balance laws:

$$\begin{cases}\frac{\partial U}{\partial t}+\frac{\partial f(U)}{\partial x} & = & -\left(\frac{\partial g(U)}{\partial y}\right)_{j} \quad \text{in } D_{i,j}^{n},\\ U(t^{n},x,y) & = & U^{n},\end{cases}$$

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An alternative approach

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$$D^n_{i,j} = \left\{ (t,x,y)/t^n \leq t \leq t^{n+\frac{1}{2}}, y^n_{j-\frac{1}{2}} \leq y \leq y^n_{j+\frac{1}{2}}, \sigma^n_i(t) \leq x \leq \sigma^n_{i+1}(t) \right\},$$

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Coupled Set of balance laws:

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} & = & -\left(\frac{\partial g(U)}{\partial y}\right)_j \quad \text{in } D^n_{i,j}, \\ U(t^n,x,y) & = & U^n, \end{cases} \quad \rightarrow \quad U^{n+\frac{1}{2}}_{i,j},$$

$$\begin{cases} \frac{\partial U}{\partial t} + \frac{\partial g(U)}{\partial y} & = & -\left(\frac{\partial f(U)}{\partial x}\right)_i \quad \text{in } D^{n+\frac{1}{2}}_{i,j}, \\ U(t^{n+\frac{1}{2}},x,y) & = & U^{n+\frac{1}{2}}, \end{cases} \quad \rightarrow \quad U^{n+1}_{i,j}.$$

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- Difficulties in accurately computing gradients near the interfaces.

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- The Buckley–Leverett equation with gravity:

$$u_t + \left(\frac{u^2}{u^2 + (1-u)^2} \right)_x + \left(\frac{u^2(1 - 5(1-u)^2)}{u^2 + (1-u)^2} \right)_y = 0.$$

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- Euler's system of compressible gas dynamics, hydrogen energy and different problems in Chemical Engineering, Biology and Medicine.

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- Advance considerably in the theory of the balance laws.

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Thanks!