Field and Order Axioms of Real Numbers in Agda

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Abstract

The introduction of real numbers can be done in different ways. In this talk, from an axiomatic construction, we formalize the real numbers and some of their properties in the proof assistant Agda.

"It is possible to construct the real number system in an entirely rigorous manner, starting from careful statements of a few of basic principles of set theory."[1]

[1] M. Rosenlicht (1968). Introduction to Analysis, p. 15.

"It is possible to construct the real number system in an entirely rigorous manner, starting from careful statements of a few of basic principles of set theory."[1]

"... assume certain basic properties (or axioms) of the real numbers system, all of which are in complete agreement with our intuition and all of which can be proved easily in the course of any rigorous construction of the system."[1]

[1] M. Rosenlicht (1968). Introduction to Analysis, p. 15.

Constant, Relationships and Basic Functions

postulate

$$\mathbb{R} : Set$$

$$r_0 : \mathbb{R}$$

$$r_1 : \mathbb{R}$$

$$\_+\_ : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$$

$$-\_ : \mathbb{R} \to \mathbb{R}$$

$$\_*\_ : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$$

$$\_^{-1} : \mathbb{R} \to \mathbb{R}$$

$$\_>\_ : \mathbb{R} \to \mathbb{R} \to Set$$

The Axioms

The field axioms, the order axioms and completeness axiom (also called the least-upper-bound axiom).

Field and Order Axioms of Real Numbers in Agda

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Field Axioms

According to the mathematical convention, it is called a field a set of two defined functions $(+, \,.)$ and also satisfy the axioms.

The Field Axioms

For all $\mathrm{a}, \mathrm{b}, \mathrm{c}$ in $\mathbb{R}$,

- Commutativity
  $a + b = b + a$,
  $a \,.\, b = b \,.\, a$.

[2] H. L. Royden and P. M. Fitzpatrick (2010). Real Analysis, p. 8.

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- Existence of Neutral Elements
  $a + 0 = a$,
  $a \,.\, 1 = a$.

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  $a + 0 = a$,
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- Existence of Additive and Multiplicative Inverses
  $a + (-a) = 0$,
  $a \,.\, a^{-1} = 1. \; (a \neq 0)$.

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- The Nontriviality Assumption[2]
  $1 \neq 0$

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  $a + (-a) = 0$,
  $a\,.\,a^{-1} = 1.\ (a \neq 0)$.
- The Nontriviality Assumption[2]
  $1 \neq 0$
- Distributivity
  $a\,.\,(b + c) = a\,.\,b + a\,.\,c$.

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The Field Axioms in Agda

```
postulate
    +comm : (x y : ℝ)   → x + y       ≡ y + x
    +asso : (x y z : ℝ) → x + y + z   ≡ x + (y + z)
    +neut : (x : ℝ)     → x + r₀      ≡ x
    +inve : (x : ℝ)     → x + (-x)    ≡ r₀
    *comm : (x y : ℝ)   → x * y       ≡ y * x
    *asso : (x y z : ℝ) → x * y * z   ≡ x * (y * z)
    *neut : (x : ℝ)     → x * r₁      ≡ x
    *inve : (x : ℝ)     → ¬ (x ≡ r₀)  → x * (x ⁻¹) ≡ r₁
    1≢0   : ¬ (r₁ ≡ r₀)
    dist  : (x y z : ℝ) → x * (y + z) ≡ x * y + x * z
```

Equality[3]

- *data* $\_\equiv\_ : \mathbb{R} \rightarrow \mathbb{R} \rightarrow$ *Set where*
  refl $: \{x : \mathbb{R}\} \rightarrow x \equiv x$

[3] Adapted from Ana Bove and Peter Dybjer (2009). Dependent Types at Work, p. 23.

Equality[3]

- $\textit{data}\ \_\equiv\_ : \ \mathbb{R} \to \mathbb{R} \to \textit{Set where}$

  $\textit{refl} : \ \{x : \mathbb{R}\} \to x \equiv x$

- Symmetry

  $\textit{sym} \ : \ \{x\ y \ : \ \mathbb{R}\} \ \to \ x \ \equiv \ y \ \to \ y \ \equiv \ x$

  $\textit{sym refl} \ = \ \textit{refl}$

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Equality[3]

- $data\ \_\equiv\_ :\ \mathbb{R} \to \mathbb{R} \to Set\ where$
  $refl :\ \{x : \mathbb{R}\} \to x \equiv x$
- Symmetry
  $sym\ :\ \{x\ y\ :\ \mathbb{R}\}\ \to\ x\ \equiv\ y\ \to\ y\ \equiv\ x$
  $sym\ refl\ =\ refl$
- Transitive
  $trans\ :\ \{x\ y\ z\ :\ \mathbb{R}\}\ \to\ x\ \equiv\ y\ \to\ y\ \equiv\ z\ \to\ x\ \equiv\ z$
  $trans\ refl\ refl\ =\ refl$

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- The Rule of $\equiv -$*elimination*[4]

$$subst\ :\ (P\ :\ \mathbb{R}\ \rightarrow\ Set) \rightarrow\ \{x\ y\ :\ \mathbb{R}\} \rightarrow x \equiv y\ \rightarrow P\ x \rightarrow P\ y$$

$$subst\ P\ refl\ Px\ =\ Px$$

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- The Rule of $\equiv -elimination$[4]

$$subst : (P : \mathbb{R} \rightarrow Set) \rightarrow \{x\ y : \mathbb{R}\} \rightarrow x \equiv y \rightarrow P\ x \rightarrow P\ y$$
$$subst\ P\ refl\ Px = Px$$

- The Rule of $\equiv -elimination$[4]

$$subst_2 : (P : \mathbb{R} \rightarrow \mathbb{R} \rightarrow Set) \rightarrow \{x_1\ x_2\ y_1\ y_2 : \mathbb{R}\} \rightarrow x_1 \equiv x_2 \rightarrow y_1 \equiv y_2 \rightarrow P\ x_1\ y_1 \rightarrow P\ x_2\ y_2$$
$$subst_2\ P\ refl\ refl\ h = h$$

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Example

$cancel : \{x\ y\ z\ : \mathbb{R}\} \rightarrow x + z \equiv y + z \rightarrow x \equiv y$

$cancel\ \{x\}\ \{y\}\ \{z\}\ h\ =$

Example

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$cancel\ \{x\}\ \{y\}\ \{z\}\ h =$

$x \quad \equiv\langle\ sym\,(+neut\ x)\ \rangle$ (1)

(3)

(5)

(7)

$y \quad \blacksquare$ (8)

Example

$cancel : \{x\ y\ z : \mathbb{R}\} \to x + z \equiv y + z \to x \equiv y$
$cancel\ \{x\}\ \{y\}\ \{z\}\ h =$

$x \quad \equiv\langle\ sym\,(+neut\ x)\ \rangle$ (1)

$x + r_0 \quad \equiv\langle\ subst\,(\lambda\, w \to (x + r_0) \equiv (x + w))$ (2)
$(sym\,(+inve\, z))\ refl\ \rangle$

(3)

(5)

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Example

$cancel \;:\; \{x\; y\; z \;:\; \mathbb{R}\} \;\to\; x + z \;\equiv\; y + z \;\to\; x \;\equiv\; y$

$cancel\; \{x\}\; \{y\}\; \{z\}\; h \;=$

x	$\equiv\langle$	$sym\,(+neut\;x)\;\rangle$	(1)
$x + r_0$	$\equiv\langle$	$subst\,(\lambda\, w \;\to\; (x + r_0) \;\equiv\; (x + w))$	(2)
		$(sym\,(+inve\,z))\;refl\;\rangle$	
$x + (z + (-z))$	$\equiv\langle$	$sym\,(+asso\;x\;z\;(-z))\;\rangle$	(3)
			(5)
			(7)
y	$\blacksquare$		(8)

Example

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$$x + (z + (-z)) \quad \equiv\langle\ sym\,(+asso\ x\ z\ (-z))\ \rangle \qquad (3)$$

$$(x + z) + (-z) \quad \equiv\langle\ subst\,(\lambda\, w\ \to\ (x + z) + (-z)\ \equiv\ w + (-z))h\ refl\ \rangle \qquad (4)$$

(5)

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$$y + (z + (-z)) \equiv\langle\ subst\,(\lambda\, w \to y + (z + (-z)) \equiv y + w)\ (+inve\ z)\ refl\ \rangle \quad (6)$$

$$(7)$$

$$y\ \blacksquare \quad (8)$$

Example

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$$cancel\ \{x\}\ \{y\}\ \{z\}\ h =$$

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The Order Axioms

For all a, b, c in $\mathbb{R}$,

- Asymmetry
  If $a > b$ *then* $b \not> a$.

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  If $c > 0$ *and* $a > b$ *then*
  $c \,.\, a > c \,.\, b$.

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For all a, b, c in $\mathbb{R}$,

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- Transitivity
  If $a > b$ *and* $b > c$ *then* $a > c$.

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  If $a > b$ *then* $c + a > c + b$.
  If $c > 0$ *and* $a > b$ *then*
  $c \,.\, a > c \,.\, b$.
- Transitivity
  If $a > b$ *and* $b > c$
  then $a > c$.
- Trichotomy
  One and only one of the following statements is true:
  $a > b$, $a = b$, $a < b$.

Field and Order Axioms of Real Numbers in Agda

The Order Axioms in Agda

```
postulate
   >asym : {x y : ℝ}   → x > y → ¬ (y > x)
  >trans : {x y z : ℝ} → x > y → y > z → x > z
   +cong : {x y z : ℝ} → x > y → z + x > z + y
   *cong : {x y z : ℝ} → z > r₀ → x > y
                        → z * x > z * y
trichotomy : (x y : ℝ) → (x > y) ∨ (x ≡ y) ∨ (x < y)
```

The Archimedean Axiom[5]

$$\forall\, a \in \mathbb{R}.\ \exists\, n \in \mathbb{N}.\ \ a < inj(n),$$

```
where
  inj : ℕ → ℝ
```

[5] Alberto Ciaffaglione and Pietro Di Gianantonio (2010). Types for Proofs and Programs, A tour with constructive real numbers, p. 43.

The Archimedean Axiom[5]

$$\forall\, a \in \mathbb{R}.\ \exists\, n \in \mathbb{N}.\ \ a < inj(n),$$

where

$inj : \mathbb{N} \to \mathbb{R}$

The Archimedean Axiom in Agda

```
postulate
  archimedean : (x : ℝ) → ∃ₙ (λ n → ℕ2ℝ n > x)
```

[5] Alberto Ciaffaglione and Pietro Di Gianantonio (2010). Types for Proofs and Programs, A tour with constructive real numbers, p. 43.

Example

$x{+}1{>}x : (x : \mathbb{R}) \rightarrow x + r_1 > x$

$x{+}1{>}x \;\; x =$ $))$

[7] $cancel : \{a\, b\, c : \mathbb{R}\} \rightarrow a + c > b + c \rightarrow a > b$

Example

$x{+}1{>}x \ : \ (x \ : \ \mathbb{R}) \ \rightarrow \ x + r_1 \ > \ x$

$x{+}1{>}x \quad x \ = \ p_1{-}helper$ $\qquad$))

[7] $cancel \ : \ \{a\, b\, c \ : \ \mathbb{R}\} \ \rightarrow \ a + c > b + c \ \rightarrow \ a > b$

Example

$x{+}1{>}x \ : \ (x \ : \ \mathbb{R}) \ \rightarrow \ x + r_1 \ > \ x$

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$x{+}1{>}x \ : \ (x \ : \ \mathbb{R}) \ \rightarrow \ x + r_1 \ > \ x$

$x{+}1{>}x \quad x \ = \ p_1{-}helper \qquad))$

where

$p_1{-}helper \ : \ r_1 + x \ > \ x \ \rightarrow \ x + r_1 \ > \ x$

$p_1{-}helper \ h \ = \ subst_2 \, (\lambda \, t_1 \, t_2 \ \rightarrow \ t_1 \ > \ t_2) \, (comm \, r_1 \, x) \, refl \, h$

[7] $cancel \ : \ \{a \, b \, c \ : \ \mathbb{R}\} \ \rightarrow \ a + c > b + c \ \rightarrow \ a > b$

Example

$x{+}1{>}x \ : \ (x \ : \ \mathbb{R}) \ \rightarrow \ x + r_1 \ > \ x$
$x{+}1{>}x \ \ x \ = \ p_1{-}helper \ (cancel^{7} \qquad\qquad))$

where

$p_1{-}helper \ : \ r_1 + x \ > \ x \ \rightarrow \ x + r_1 \ > \ x$
$p_1{-}helper \ h \ = \ subst_2 \ (\lambda \ t_1 \ t_2 \ \rightarrow \ t_1 \ > \ t_2) \ (comm \ r_1 \ x) \ refl \ h$

[7] $cancel \ : \ \{a \ b \ c \ : \ \mathbb{R}\} \ \rightarrow \ a + c > b + c \ \rightarrow \ a > b$

Example

$x{+}1{>}x\ :\ (x\ :\ \mathbb{R})\ \rightarrow\ x + r_1\ >\ x$
$x{+}1{>}x\ \ x\ =\ p_1{-}helper\,(cancel^{7}\,(p_2{-}helper\ \ ))$

where

$p_1{-}helper\ :\ r_1 + x\ >\ x\ \rightarrow\ x + r_1\ >\ x$
$p_1{-}helper\ h\ =\ subst_2\,(\lambda\, t_1\, t_2\ \rightarrow\ t_1\ >\ t_2)\,(comm\, r_1\, x)\, refl\, h$

$p_2{-}helper\ :\ r + (x + (-x))\ >\ r_0\ \rightarrow\ (r_1 + x) + (-x)\ >\ x + (-x)$
$p_2{-}helper\ h\ =\ subst_2\,(\lambda\, t_1\, t_2\ \rightarrow\ t_1\ >\ t_2)\,(sym\,(asso\, r_1\, x\,(-x)))\,(sym\,(inve\, x))\, h$

[7] $cancel\ :\ \{a\, b\, c\ :\ \mathbb{R}\}\ \rightarrow\ a + c > b + c\ \rightarrow\ a > b$

Example

$x{+}1{>}x\ :\ (x\ :\ \mathbb{R}) \rightarrow x + r_1 > x$
$x{+}1{>}x\ \ x\ =$
$p_1{-}helper\,(cancel\,(p_2{-}helper \qquad))))$

where

Example

$x{+}1{>}x \ : \ (x \ : \ \mathbb{R}) \rightarrow x + r_1 \ > \ x$

$x{+}1{>}x \ \ x \ =$

$p_1 - helper\,(cancel\,(p_2 - helper\ (p_3 - helper \qquad\qquad))))$

`where`

$p_3 - helper \ : \ r_1 + r_0 \ > \ r_0 \ \rightarrow \ r_1 + (x + (-\,x)) \ > \ r_0$

$p_3 - helper\ h \ = \ subst_2\,(\lambda\, t_1\, t_2 \ \rightarrow \ t_1 \ > \ t_2)\,(subst\,(\lambda\, w \ \rightarrow$
$r_1 + r_0 \ \equiv \ r_1 + w)\,(\equiv{-}sym\,(+{-}inve\, x))\ refl)\ refl\ h$

Example

```
x+1>x : (x : ℝ) → x + r₁ > x
x+1>x x =
p₁−helper (cancel (p₂−helper (p₃−helper (p₄−helper          ))))

  where
      p₃−helper : r₁ + r₀ > r₀ → r₁ + (x + (− x)) > r₀
      p₃−helper h = subst₂ (λ t₁ t₂ → t₁ > t₂) (subst (λ w →
                        r₁ + r₀ ≡ r₁ + w) (≡−sym (+−inve x)) refl) refl h

      p₄−helper : r₁ > r₀ → r₁ + r₀ > r₀
      p₄−helper r₁>r₀ : subst₂ (λ t₁ t₂ → t₁ > t₂) (sym (neut r₁))
                                                    refl r₁>r₀
```

Example

$x{+}1{>}x \;:\; (x \;:\; \mathbb{R}) \to x + r_1 \;>\; x$
$x{+}1{>}x \;\; x \;=$
$p_1{-}helper\,(cancel\,(p_2{-}helper\;(p_3{-}helper\;(p_4{-}helper(r_1{>}r_0)))))$

where

$p_3{-}helper \;:\; r_1 + r_0 \;>\; r_0 \;\to\; r_1 + (x + (-\,x)) \;>\; r_0$
$p_3{-}helper\; h \;=\; subst_2\,(\lambda\, t_1\, t_2 \;\to\; t_1 \;>\; t_2)\,(subst\,(\lambda\, w \;\to$
$r_1 + r_0 \;\equiv\; r_1 + w)\,(\equiv{-}sym\,(+{-}inve\, x))\; refl)\; refl\; h$

$p_4{-}helper \;:\; r_1 \;>\; r_0 \;\to\; r_1 + r_0 \;>\; r_0$
$p_4{-}helper\; r_1{>}r_0 \;:\; subst_2\,(\lambda\, t_1\, t_2 \;\to\; t_1 \;>\; t_2)\,(sym\,(neut\; r_1))$
$refl\; r_1{>}r_0$

Examples

To click on Example

- Example 1

Examples

To click on Example

- Example 1
- Example 2

Examples

To click on Example

- Example 1
- Example 2
- Example 3

Examples

To click on Example

- Example 1
- Example 2
- Example 3
- Example 4

Examples

To click on Example

- Example 1
- Example 2
- Example 3
- Example 4
- Example 5

Future Work

- Completeness Axiom
  Axiomatize in Agda, the completeness property and define previously in Agda the concepts of upper bound, set of upper bounds and the least upper bound.

[6] http://www1.eafit.edu.co/asr/publications.html

Future Work

- Completeness Axiom
  Axiomatize in Agda, the completeness property and define previously in Agda the concepts of upper bound, set of upper bounds and the least upper bound.

- Automate the processes of demonstration using as reference the work of Ana Bove, Peter Dybjer, and Andrés Sicard-Ramírez[6]

[6] http://www1.eafit.edu.co/asr/publications.html

Future Work

- The Archimedean Axiom in Coq
  The Coq proof assistant, presented in its standard library[7] a long version of the Archimedean property:

  $$\forall\, a \in \mathbb{R}.\ \exists\, n \in \mathbb{N}.\ a < n \ \wedge\ n - a \leq 1.$$

  With this axiom the property $x < x + 1$ is proved. In our case we did with the axioms of field and order.
  Would be an interesting study to analyze the properties which demonstrates Coq using the Archimedean property and do demonstrations with the short version of this axiom.

[7] The Coq Proof Assistant (8.4pl4). https://coq.inria.fr/distrib/current/stdlib/Coq.Reals.Raxioms.html

The Completeness Axiom

"A nonempty set E of real numbers is said to be **bounded above** provided there is a real number b such that $x \leq b$ for all $x \in E$: the number b is called an **upper bound** for E. Similarly, we define what it means for a set to be **bounded below** and for a number to be a **lower bound** for a set. A set that is bounded above need not have a largest member. But the next axiom asserts that it does have a smallest upper bound.

The Completeness Axiom "Let E be a nonempty set of real numbers that is bounded above. Then among the set of upper bounds for E there is a smallest, or least, upper bound."[8]

[8] H. L. Royden and P. M. Fitzpatrick (2010). Real Analysis. p. 9.

Field and Order Axioms of Real Numbers in Agda

Definition

- $\forall a, b \in \mathbb{R}$, the relationship is introduced $" > "$ and $" < "$ Either the expression:

$$a > b \text{ or } b < a.$$

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Definition

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- Where $a - b > 0$, also:

$$a \geq b \text{ or } b \leq a.$$

- That would be equivalent to: $a > b$ or $a = b$.

Reasoning Equational[9]

- $\_ \equiv\langle \_ \rangle \_ : \forall x \; \{y \; z\} \to x \equiv y \to y \equiv z \to x \equiv z$
  $\_ \equiv\langle x \equiv y \rangle \;\; y \equiv z \; = \; \mathit{trans} \;\; x \equiv y \;\; y \equiv z$

[9] Mu, S.-C., Ko, H.-S. and Jansson, P. (2009). Algebra of Programming in Agda: Dependent Types for Relational Program Derivation. Journal of Functional Programming 19.5, pp. 545-579.

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- $\_\blacksquare\ : \forall x \rightarrow x \equiv x$
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- $\_\blacksquare\ :\ \forall x \to x \equiv x$
  $\_\blacksquare\ \ \_ = refl$

- Injection

  $\mathbb{N}2\mathbb{R} : \mathbb{N} \to \mathbb{R}$
  $\mathbb{N}2\mathbb{R}\ (zero) = r_0$
  $\mathbb{N}2\mathbb{R}\ (succ\ n) = \mathbb{N}2\mathbb{R}\ n + r_1$

[9] Mu, S.-C., Ko, H.-S. and Jansson, P. (2009). Algebra of Programming in Agda: Dependent Types for Relational Program Derivation. Journal of Functional Programming 19.5, pp. 545-579.

Bottom and Existential

- Bottom[10]

  data $\bot$: *Set where*

  $\bot-$*elim* : $\{A : Set\} \to \bot \to A$

  $\bot-$*elim* ( )

[10] Ana Bove and Peter Dybjer (2009). Dependent Types at Work, p. 8

[11] Ana Bove and Peter Dybjer (2009). Dependent Types at Work. p. 9

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- Bottom[10]

  data $\bot$: *Set where*

  $\bot-$*elim* : $\{A : Set\} \rightarrow \bot \rightarrow A$

  $\bot-$*elim* ( )

- $\neg\_ : Set \rightarrow Set$

  $\neg A = A \rightarrow \bot$

[10] Ana Bove and Peter Dybjer (2009). Dependent Types at Work, p. 8

[11] Ana Bove and Peter Dybjer (2009). Dependent Types at Work. p. 9

Field and Order Axioms of Real Numbers in Agda

Bottom and Existential

- Bottom[10]

  $\textbf{data} \perp \; : \; \mathit{Set\ where}$

  $\quad \perp\!-\mathit{elim} \; : \; \{A \; : \; \mathit{Set}\} \; \rightarrow \; \perp \; \rightarrow \; A$

  $\quad \perp\!-\mathit{elim} \; (\,)$

- $\neg\_ \; : \; \mathit{Set} \; \rightarrow \; \mathit{Set}$

  $\neg \, A \; = \; A \; \rightarrow \perp$

- Existential[11]

  $\textbf{data} \; \exists_r \; (P \; : \; \mathbb{R} \; \rightarrow \; \mathit{Set}) \; : \; \mathit{Set\ where}$

  $\quad \mathit{exist} \; : \; (x \; : \; \mathbb{R}) \; \rightarrow \; P \, x \; \rightarrow \; \exists_r \; P$

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[11] Ana Bove and Peter Dybjer (2009). Dependent Types at Work. p. 9

Conjuntion[12]

- $\textbf{data}\ \_\wedge\_\ (A\ B\ :\ Set)\ :\ Set\ where$

  $\_,\_\ :\ A\ \rightarrow\ B\ \rightarrow\ A\ \wedge\ B$

[12] Ana Bove and Peter Dybjer (2009). Dependent Types at Work, pp. 18-19.

Conjuntion[12]

- $\mathbf{data}\ \_\wedge\_\ (A\ B\ :\ Set)\ :\ Set\ where$
  $\_,\_\ :\ A\ \rightarrow\ B\ \rightarrow\ A\ \wedge\ B$

- $proj_1\ :\ \forall \{A\ B\}\ \rightarrow\ A\ \wedge\ B\ \rightarrow\ A$
  $proj_1\ (a\ ,\ \_)\ =\ a$

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Conjuntion[12]

- $\textbf{data}\ \_\wedge\_\ (A\ B\ :\ Set)\ :\ Set\ where$
  $\_,\_\ :\ A\ \rightarrow\ B\ \rightarrow\ A\ \wedge\ B$

- $proj_1\ :\ \forall\{A\ B\}\ \rightarrow\ A\ \wedge\ B\ \rightarrow\ A$
  $proj_1\ (a\ ,\ \_)\ =\ a$
- $proj_2\ :\ \forall\{A\ B\}\ \rightarrow\ A\ \wedge\ B\ \rightarrow\ B$
  $proj_2\ (\_\ ,\ b)\ =\ b$

[12] Ana Bove and Peter Dybjer (2009). Dependent Types at Work, pp. 18-19.

Disjunction[13]

- **data** $\_\vee\_$ $(A\ B\ :\ Set)\ :\ Set\ where$
  $inj_1\ :\ A\ \rightarrow\ A\ \vee\ B$
  $inj_2\ :\ B\ \rightarrow\ A\ \vee\ B$

[13] Ana Bove and Peter Dybjer (2009). Dependent Types at Work, pp. 19-20.

Disjunction[13]

- $\textbf{data}\ \_\lor\_\ (A\ B\ :\ Set)\ :\ Set\ where$
  $inj_1\ :\ A\ \to\ A\ \lor\ B$
  $inj_2\ :\ B\ \to\ A\ \lor\ B$

- $case\ :\ \forall\{A\ B\}\ \to\ \{C\ :\ Set\}\ \to\ (A\ \to\ C)\ \to$
  $(B\ \to\ C)\ \to\ A\ \lor\ B\ \to\ C$
  $case\ f\ g\ (inj_1\ a)\ =\ f\ a$
  $case\ f\ g\ (inj_2\ b)\ =\ g\ b$

[13] Ana Bove and Peter Dybjer (2009). Dependent Types at Work, pp. 19-20.