Girard's Paradox

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Introduction

Brief history [Sørensen and Urzyczyn, 2006]

- Russell's type theory (1903).
- Simply typed lambda-calculus by Church (1940).
- Curry-Howard correspondence (1934-1969).
- Martin Löf's intuitionistic type theory (1971).
- Girard's paradox (1972).
- Barendregt's λ-cube (1991).
- Pure Type Systems by Barendregt (1992).

Simply type theory

We start with two infinite sets **B** and V.

- **Types: T**:=**B**|**T**→**T**.
- **Terms:** Λ:=V|ΛΛ|λV:**T**.Λ.
- Application: if $M : \sigma \to \tau$ and $N : \sigma$, then $MN : \tau$.
- Abstraction: if $P : \tau$, then $\lambda x : \sigma.P : \sigma \to \tau$.

β-reduction

$$(\lambda x : \sigma.M)N \to_\beta M[x := N]$$

- **Typing à la Church**: we have terms with type information in the λ-abstraction. Terms have unique types.
- **Typing à la Curry**: we assign types to untyped λ-terms. Terms do not have unique types.

Examples:

- $\lambda x : \alpha.\lambda y : (\beta \to \alpha) \to \alpha.y(\lambda z : \beta.x)$

has only a type:

$$\alpha \to ((\beta \to \alpha) \to \alpha) \to \alpha$$

- $\lambda x.\lambda y.y(\lambda z.x)$

can be given the types

$$\alpha \to ((\beta \to \alpha) \to \alpha) \to \alpha,$$
$$(\alpha \to \alpha) \to ((\beta \to \alpha \to \alpha) \to \gamma) \to \gamma,$$
$$\ldots$$

Formulas-as-Type

There are three readings of a judgement $a : A$ [Palmgren, 2013].

A set	$a : A$	
A is a set	a is an element of the set A	A is non-empty (or inhabited)
A is a proposition	a is a proof (construction) of the proposition A	A is true
A is a problem (task)	a is a method of solving the problem (doing the task) A	A is solvable

Pure type systems

A pure type system [Sørensen and Urzyczyn, 2006] (PTS) is determined by a triple (S, A, R) with:

1. S the set of *sorts*.
2. $A \subseteq S \times S$ the set of *axioms*.
3. $R \subseteq S \times S \times S$ the set of *rules*.

If $s_2 = s_3$, a rule (s_1, s_2, s_3) is abbreviated (s_1, s_2).
Pseudoterms:

$$\text{T:=S} \mid \text{Var} \mid (\Pi\text{Var:T.T}) \mid (\lambda\ \text{Var:T.T}) \mid \text{TT}.$$

where Var is a infinite set of *variables*.

We have the following derivation rules:

Derivation rules for a PTS

(Ax) $\varnothing \vdash s_1 : s_2, \quad \text{when } (s_1, s_2) \in \mathcal{A}$

(Var) $$\frac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A} \quad (x \notin \mathrm{dom}(\Gamma))$$

(Prod) $$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.\, B) : s_3} \quad ((s_1, s_2, s_3) \in \mathcal{R})$$

(Abs) $$\frac{\Gamma, x{:}A \vdash B : C \qquad \Gamma \vdash (\Pi x{:}A.\, C) : s}{\Gamma \vdash (\lambda x{:}A.\, B) : (\Pi x{:}A.\, C)}$$

(App) $$\frac{\Gamma \vdash A : (\Pi x{:}B.\, C) \qquad \Gamma \vdash D : B}{\Gamma \vdash (AD) : C[x := D]}$$

(Weak) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash C : s}{\Gamma, x{:}C \vdash A : B} \quad (x \notin \mathrm{dom}(\Gamma))$$

(Conv) $$\frac{\Gamma \vdash A : B \qquad \Gamma \vdash B' : s}{\Gamma \vdash A : B'} \quad (B =_\beta B')$$

System λHOL

System λHOL is the PTS with sorts $\{*, \square, \triangle\}$, axioms $* : \square$ and $\square : \triangle$, and rules $\{(*, *), (\square, *), (\square, \square)\}$.
In λHOL it is possible to code the minimal intuitionistic predicate logic, which only has implication and universal quantification as logical operations [Nederpelt and Geuvers, 2014].

We code the implication $A \Rightarrow B$ as the function type $A \rightarrow B$, where $A \rightarrow B := \Pi x : A.B$ if x does not occur free in B.
We code the universal quantification $\forall_{x \in X}(P(x))$ as the type $\Pi x : S.Px$.

Furthermore, in λHOL it is possible to code the intuitionistic logic, even the high order intuitionisticc logic[Barendregt et al., 2013].

Definability of logical connectives

$\bot := \Pi\alpha : *.\alpha$
$\sigma \wedge \tau := \Pi\alpha * .(\sigma \to \tau \to \alpha) \to \alpha$
$\sigma \vee \tau := \Pi\alpha * .(\sigma \to \alpha) \to (\tau \to \alpha) \to \alpha$
$\exists := \Pi\beta * .(\Pi\alpha * .\sigma \to \beta) \to \beta$
Let A be a type, we define the Leibniz equality:
$a =_A b := \Pi P : A \to *.(Pa) \to (Pb)$ [Nederpelt and Geuvers, 2014].

The systems λU^-, λU

1. System λU is a PTS with sorts $\{*, \Box, \triangle\}$, axioms $* : \Box$ and $\Box : \triangle$, and rules $\{(*, *), (\Box, *)\}, (\Box, \Box), (\triangle, *), (\triangle, \Box)\}$.
2. System λU^- is the fragment of the system λU without rule $(\triangle,*)$.

In λU^- we obtain higher order logic over polymorphic domains.

λU allows quantification over all types.

A PTS λS is called inconsistent if there exists a term M such that $\vdash_{\lambda S} M : \bot$ [Geuvers, 1993].

Theorem 1 (Girard's Paradox)

Systems λU y λU^{-} are inconsistent.

1. λU can't be used as a logic.
2. Every type in λU^{-} and λU is inhabited.
3. There exists a term in λU^{-} without normal form.

Corolario 1.1 ("Type is not a type)

$\lambda *$ is inconsistent.

1. Consider a paradox in naive set theory

The colection of all ordinal numbers is not a set. (Burali-Forti’s Paradox) [Hurkens, 1995].

2. Abstract version of the paradox

Let $\mathcal{U}$ be a set, $\sigma : \mathcal{U} \to \mathscr{P}\mathcal{U}$ and $\tau : \mathscr{P}\mathcal{U} \to \mathcal{U}$ such that for each X in $\mathscr{P}\mathcal{U}$, $\sigma\tau X = \{\tau\sigma x | x\ is\ a\ element\ of\ X\}$ [Hurkens, 1995].
Let $\approx$ be the least equivalence relation on $\mathcal{U}$ such that for each x in $\mathcal{U}$, $x \approx \tau\sigma x$. Define $\in$ a relaction on $\mathcal{U}$ defined as follows

$$y \in x \text{ if and only if } y \approx z \text{ for some } z \text{ in } \sigma x.$$

Let $\Delta := \tau\{x|x \notin x\}$. Then for each y in $\mathcal{U}$, $y \in \Delta$ if and only if $y \notin y$.
Take $y = \Delta$[Hurkens, 1995].

3. Formalizing

In λHOL, can be formalized the preceding derivation of a contradiction in the context
$\mathcal{U} : \square$,
$\sigma : \mathcal{U} \to \mathscr{P}\mathcal{U}$, with $\mathscr{P}\mathcal{U} := \mathcal{U} \to *$,
$\tau : \mathscr{P}\mathcal{U} \to \mathcal{U}$,
$P : (\Pi X : \mathscr{P}\mathcal{U}.(\sigma(\tau X)) =_{\mathscr{P}\mathcal{U}} \lambda u : \mathcal{U}.\exists x : \mathcal{U}.((Xx) \wedge u =_{\mathcal{U}} (\tau(\sigma x)))$,
where $=_A$ is the Leibniz equality.

4. The paradox in λU^-

Define $\mathcal{U}$, σ y τ en λU^- such that P is inhabited.
Hence, we obtain a term with type $\bot$.

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