Towards a Paraconsistent Type Theory

Juan Carlos Agudelo-Agudelo (joint work with Andrés Sicard-Ramírez)

Institute of Mathematics University of Antioquia - UdeA Medellín - Colombia

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"Intuitionistic logic, sometimes more generally called constructive logic, refers to systems of symbolic logic that differ from the systems used for classical logic by more closely mirroring the notion of constructive proof... Formalized intuitionistic logic was originally developed by Arend Heyting to provide a formal basis for Brouwer's programme of intuitionism." [Wikipedia contributors, 2019]



A natural deduction system for Intuitionistic Logic (IL) ([van Dalen, 2013]):

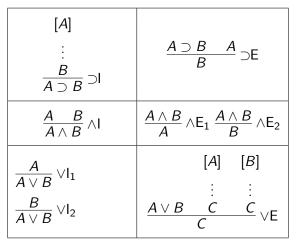


Table: Rules for propositional connectives in IL



Definition of negation: $\neg A \stackrel{\text{\tiny def}}{=} A \supset \bot$.



Table: Bottom elimination rule



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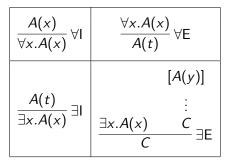


Table: Rules for quantifiers in IL



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- The following are not theorems of IL:
 - Tertium Non Datur: $A \lor \neg A$.
 - Doble negation elimination: $\neg \neg A \supset A$.
 - Reductio ad adsurdum: $(\neg A \supset B) \supset ((\neg A \supset \neg B) \supset A)$.



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- Several intuitionistic theories, for instance Heyting arithmetic, satisfy the existence property: if ∃x.A(x) is a theorem of the theory, then A(t) is a theorem of the theory for some term t.
- IL satisfies substitution by equivalents.



The Brouwer-Heyting-Kolmogorov (BHK) interpretation (or proof interpretation) for IL ([Troelstra and van Dalen, 1988]):

- a proves A ∧ B if a is a pair (b, c) such that b proves A and c proves B.
- a proves A ∨ B if a is a pair (b, c) such that b is a natural number and if b = 0 then c proves A, otherwise c proves B.
- a proves A ⊃ B if a is a construction that converts any proof p of A into a proof a(p) of B.
- no *a* proves \perp .



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In order to deal with the quantifiers, it is assumed that some domain D of objects is given.

- a proves $\forall x.A(x)$ if a is a construction such that, for each $b \in D$, a(b) proves $A(\overline{b})$.
- a proves $\exists x.A(x)$ if a is a pair $\langle b, c \rangle$ such that $b \in D$ and c proves $A(\overline{b})$.













A brief historical review:

- (1908) Bertran Russel's proposed a ramified theory of types.
- (1920s) Leon Chwistek and Frank P. Ramsey proposed an unramified type theory, now known as theory of simple types (or simple theory of types).
- (1940) Alonso Church introduced the simply typed lambda-calculus.
- (1958) Haskell Curry establishes a correspondence between the simply typed lambda-calculus and the implicational fragment of intuitionistic logic.
- (1969) William A. Howard extended the correspondence to firts-order predicate logic, which is now known as the Curry-Howard correspondence.
- (1970s) Per Martin-Löf introduces several different versions of his theory of types.

The name "intuitionistic type theory (ITT)" is somewhat ambiguous, but usually refers to a version (or a modified version) of Martin-Löf's type theory. Because of that, "intuitinistic type theory" and "Martin-Löf's type theory" are considered synonyms.



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ITT is based on:

- Martin-Löf's analysis of the notion of "judgement" (in mathematics).
- Martin-Löf's (intuitionistic) conception of the logical connectives.
- The Curry-Howard correspondence.
- A (not precisely defined) notion of "inductive definition".



ITT extends the BHK-interpretation "to the more general setting of intuitionistic type theory and thus provides a general conception not only of what a constructive proof is, but also of what a constructive mathematical object is." ([Dybjer and Palmgren, 2016]).



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For Martin-Löf, "A judgement is an act of knowledge, for instance asserting that something holds. When reasoning mathematically we are making a sequence of judgements about mathematical objects. One kind of judgement may be to state that some mathematical statement is true, another kind of judgement may be to state that something is a mathematical object, is a set, for instance. The logical rules give a method for producing correct judgements from earlier judgements." [Palmgren, 2013, p. 10]



Martin-Löf's type theory have four basic forms of judgements ([Martin-Löf, 1984, p. 5–10]):

- A is a set (abbreviated A : set).
- **2** A and B are equal sets (abbreviated A = B).
- **3** *a* **is an element of the set** *A* (abbreviated a : A).
- a and b are equal elements of the set A (abbreviated a = b : A).



Four different interpretations for Martin-Löf's judgements forms are ([Martin-Löf, 1984, p. 5]):

A : set	a : A	
A is a set	<i>a</i> is an element of the set <i>A</i>	A is non-empty
A is a proposition	<i>a</i> is a proof (construction) of proposition <i>A</i>	A is true
A is an intention (expectation)	<i>a</i> is a method of fulfilling (realizing) the intention (ex- pectation) <i>A</i>	A is fulfillable (realizable)
A is a problem (task)	<i>a</i> is a method of solving the problem (doing the task) <i>A</i>	A is solvable

Defining a set (or a type) in Martin-Löf's type theory requires the following rules ([Martin-Löf, 1984, p. 24]):

- Formation: says that we can form a certain set from certain other sets or families of sets.
- Introduction: say what are the canonical elements (and equal canonical elements) of the set, thus giving its meaning.
- Elimination: shows how we may define functions on the set defined by the introduction rules.
- Equality/Computation: relate the introduction and elimination rules by showing how a function defined by means of the elimination rule operates on the canonical elements of the set which are generated by the introduction rules.



Intuitionistic type theory

Definition of sets related with logical connectives ([Palmgren, 2013]):

Function set:

$ \begin{array}{r} $	$ \begin{array}{c} \rightarrow \text{-introduction} \\ [x : A] \\ \vdots \\ b(x) : B \\ \overline{\lambda x.b(x) : A \rightarrow B} \end{array} $
$\frac{\rightarrow \text{-elimination}}{b: A \rightarrow B a: A}{Ap(b, a): B}$	$\rightarrow -computation$ $[x : A]$ \vdots $a : A b(x) : B$ $Ap(\lambda x.b(x), a) = b(a) : B$

Product of two sets:

$ \frac{A:set B:set}{A \times B:set} $	$\frac{a:A b:B}{(a,b):A \times B}$
$\frac{\text{$$\times$-elimination}}{\frac{c:A \times B}{\pi_1(c):A}} \frac{c:A \times B}{\pi_2(c):B}$	$\frac{a:A b:B}{\pi_1((a,b)) = a:A}$ $\frac{a:A b:B}{\pi_2((a,b)) = b:B}$



Intuitionistic type theory

Disjoint union of two sets:

$\frac{A:set}{A+B:set}$	$\frac{\begin{array}{c} +-\text{introduction} \\ \hline a:A \\ \hline \text{inl}(a):A+B \\ \hline \text{inr}(b):A+B \end{array}$
+-elimination	+-computation
$[x:A] [y:B]$ $\vdots \vdots$ $\frac{c:A+B d(x):C e(y):C}{D(c,(x)d(x),(y)e(y)):C}$	$[x:A] [y:B]$ $\vdots \vdots$ $inl(a):A+B d(x):C e(y):C$ $D(inl(a),(x)d(x),(y)e(y)) = d(a):C$ $[x:A] [y:B]$ $\vdots \vdots$ $inr(b):A+B d(x):C e(y):C$ $D(inr(b),(x)d(x),(y)e(y)) = e(b):C$



Product of a family of sets:

Π-formation	Π-introduction
[x : A]	[x : A]
:	:
A: set B(x): set	b(x):B(x)
$\Pi x : A.B(x) : set$	$\overline{\lambda x.b(x)}$: Πx : $A.B(x)$
П-elimination	П-computation
$b: \Pi x : A.B(x)$ $a: A$	[x : A]
Ap(b,a) : $B(a)$:
	a : A $b(x)$: $B(x)$
	$\overline{Ap(\lambda x.b(x),a)=b(a):B(a)}$



Disjoint union of a family of sets:

$\Sigma -formation \qquad [x : A] \\ \hline \frac{A : set B(x) : set}{\Sigma x : A.B(x) : set}$	$\Sigma\text{-introduction} \frac{a:A b:B(a)}{(a,b):\Sigma x:A.B(x)}$
Σ-elimination	Σ-computation
[x:A,y:B(x)]	[x:A,y:B(x)]
$c: \Sigma x: A.B(x) \qquad d(x,y): C((x,y))$	a:A b:B(a) d(x,y):C((x,y))
E(c,(x,y)d(x,y)):C(c)	$\overline{E((a,b),(x,y)d(x,y))} = d(a,b) : C((a,b))$
	UNVERSIDA

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Intuitionistic type theory

Empty set:

N_0 -formation	N_0 -elimination
$\overline{N_0:set}$	$\frac{c: N_0 C: set}{R_0(c): C}$



The Curry-Howard correspondence for ITT:

Logic	Type theory
Proposition	Type
Connective	Type constructor
Implication	Function type
Conjunction	Product of two types
Disjunction	Disjoint union of two types
For all	Product of a family of types
Exists	Disjoint union of a family of types
Absurdity	Empty type
Proof	Term
Provability	Inhabitation













Some objections against IL:

"... definitions of constructiveness other than that advocated by the intuitionists are conceivable. For that matter, even the small number of actual intuitionists do not completely agree about the delimination of the constructive." [Heyting, 1971, p. 10]



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"Serious objections against the use of negation in mathematics have been raised by Griss [...]. Though agreeing completely with Brouwer's basic ideas on the nature of mathematics, he contends that every mathematical notion has its origin in a mathematical construction, which can actually be carried out; if the construction is impossible, then the notion cannot be clear." [Heyting, 1971, p. 124]



Nelson's paraconsistent logic

"The justification [of the ex falso rule] in terms of constructions is not universally accepted, e.g. [Johansson, 1936] rejected the rule and formulated his so-called minimal logic, which has the same rules of intuitionistic logic with deletion of the ex falso rule." [van Dalen, 2002, p. 12]



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Some proposals to overcome the objections against IL:

- Georg F. C. Griss: eliminate negation from IL (negationless constructive mathematics).
- Ingebrigt Johansson: eliminate the *ex falso* rule from IL (minimal logic).
- **David Nelson**: define constructive rules for negation (Nelson's logic with strong (or constructive) negation).



Nelson's paraconsistent logic

A natural deduction system for Nelson's Logic N (also known as N3) is obtained by adding to the system for IL the rules ([Prawitz, 1965, p. 97]):

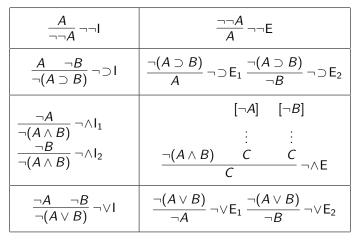


Table: Rules for negation of propositional connectives in N J. C. Agudelo-Agudelo Towards a Paraconsistent Type Theory



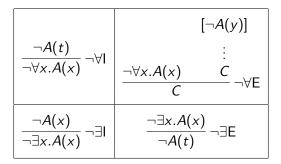


Table: Rules for negation of quantifiers in N



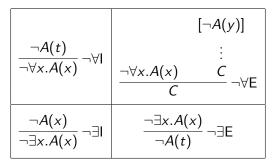


Table: Rules for negation of quantifiers in N

Intuitionistic and strong negation are connected by the additional rule:



Table: Bottom introduction rule

Under a logic L:

- A theory Γ is contradictory if there exists a formula A such that Γ⊢_L A and Γ⊢_L ¬A.
- A theory Γ is trivial if $\Gamma \vdash_L A$ for every formula A.

A paraconsistent logic is a logic that admits contradictory but non-trivial theories.



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IL and N are not paraconsistent logics.

Nelson's logic N^- (also known as N4), introduced in [Almukdad and Nelson, 1984]), can be defined by removing the bottom rules from N. The logic N^- is paraconsistent.



- The following are not theorems of N⁻:
 - Non-contradiction: $\neg (A \land \neg A)$.
 - Ex falso sequitur quodlibet: $\neg A \supset (A \supset B)$.
 - Law of contradiction: $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.
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 - Doble negation: $(\neg \neg A \supset A) \land (A \supset \neg \neg A)$.
 - De Morgan laws. $(\neg(A \land B) \supset (\neg A \lor \neg B)) \land ((\neg A \lor \neg B) \supset \neg(A \land B))$



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 - De Morgan laws. $((A \land B) \supseteq ((A) \land (B)) \land (((A) \land B)) \land ((A) \land (A) \land$

 $(\neg (A \land B) \supset (\neg A \lor \neg B)) \land ((\neg A \lor \neg B) \supset \neg (A \land B))$

N⁻ satisfies the disjunction property: ⊢_{N⁻} A ∨ B iff ⊢_{N⁻} A or ⊢_{N⁻} B.



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- N⁻ does not satisfy substitution by equivalents (however substitution is valid for equivalent formulas whose negations are also equivalents).



Extension of the BHK-interpretation, where refutation is a primitive notion and a construction c proves $\neg A$ iff c refutes A ([López-Escobar, 1972]):

- c refutes $\neg A$ iff c proves A.
- c refutes $A \wedge B$ iff c = (i, d) with i either 0 or 1 and if i = 0, then d refutes A and if i = 1 then d refutes B.
- c refutes $A \lor B$ iff c = (d, e) and d refutes A and e refutes B.
- c refutes $A \supset B$ iff c = (d, e) and d proves A and e refutes B.
- c refutes $\forall x.A(x)$ iff c = (a, d) and d refutes A(a).
- c refutes ∃x.A(x) iff c is a general method of construction such that given any individual (i.e construction) a from the species under consideration, c(a) (i.e. c applied to a) refutes A(a).







Towards a paraconsistent type theory



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Main objective: construct a type theory based on Nelson's paraconsistent logic N^- .



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Related works:

- [Wansing, 1993]: introduced a typed lambda-calculus (λ^C) where types are the propositional formulas of N⁻ and rules are based on the rules of N⁻. A formulas-as-types correspondence and a semantics for λ^C is provided. Assignation of types is not unique and issues like normalization are left open.
- [Kamide, 2010]: a different typed lambda calculus for the propositional fragment of N⁻ is provided and strong normalization for this calculus is proven.



A proposal of a paraconsistent theory of types (PTT): add opposite types to ITT, including introduction, elimination and computation rules for each type constructor.

[–] -formation	
$\frac{\underline{A:set}}{\overline{A:set}}$	
⁻ -introduction	[–] -elimination
$\frac{a:A}{a:\overline{\overline{A}}}$	$\frac{a:\overline{\overline{A}}}{a:A}$
$\frac{a:A b:\overline{B}}{(a,b):\overline{A}\to B}$	$\frac{c:\overline{A\to B}}{\pi_1(c):A}\frac{c:\overline{A\to B}}{\pi_2(c):\overline{B}}$



Towards a paraconsistent type theory

-introduction	[–] -elimination
$\frac{a:\overline{A}}{\operatorname{inl}(a):\overline{A\times B}}$ $\frac{b:\overline{B}}{\operatorname{inr}(b):\overline{A\times B}}$	$[x:\overline{A}] \qquad [y:\overline{B}]$ $\vdots \qquad \vdots$ $\frac{c:\overline{A \times B} d(x):C e(y):C}{D(c,(x)d(x),(y)e(y)):C}$
$\frac{\underline{a}:\overline{A}}{(a,b):\overline{A}+\overline{B}}$	$\frac{c:\overline{A+B}}{\pi_1(c):\overline{A}} \frac{c:\overline{A+B}}{\pi_2(c):\overline{B}}$



$\begin{array}{c c} a: A & b: \overline{B(a)} \\ \hline \end{array}$	-introduction	⁻ -elimination
$\underline{b: \overline{\Sigma x : A.B(x)}} a: A$		$[x : A, y : \overline{B(x)}]$ \vdots $\frac{c : \overline{\Pi x : A.B(x)} d(x, y) : C((x, y))}{E(c, (x, y)d(x, y)) : C(c)}$
$\lambda x.b(x):\overline{\Sigma x:A.B(x)}$	$b(x):\overline{B(x)}$	

Note: --computation rules are defined just in the same way that in ITT.

The Curry-Howard correspondence for PTT:

Logic	Type theory
Proposition	Туре
Connective	Type constructor
Implication	Function type Product of two types
Conjunction	Product of two types
Disjunction	Disjoint union of two types Product of a family of types Disjoint union of a family of types
For all	Product of a family of types
Exists	Disjoint union of a family of types
Negation	Opposite type
Proof/Refutation	Term
Provability/Refutability	Inhabitation



Towards a paraconsistent type theory

Some properties of PTT:

- Tolerance to contradictions.
- No uniqueness of types.
- Strong normalization.



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