Analysis of processes capability using the skewed normal distribution

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***Abstract*— A process quality is the ability it has to produce goods according to item specifications and it is measured through process capability indices. Real data associated to production processes do not always fit into a normal distribution, even though most industries assume normal behavior. Real process data are flexible and are influenced by various external factors requiring considering process capability indices for asymmetric distributions, or giving wrong information otherwise and preventing industry of performing what is really needed. Skewed Normal distribution is an excellent tool to fit real process data into capability indices since it can contemplate three associated behaviors regarding asymmetric distributions. A flowchart is added to explain indices calculation and the methodology is implemented in R programming language.**

Keywords— **Process capability index, Clements's percentile method, Skewed Normal distribution.**

I. Introduction

Process capability is a measurable property of a process, refers to the range within which the natural variation occurs, that is, the variation produced by random causes only [1]. The process capability analysis includes calculating process capability indices (PCIs), in order to perform continuous monitoring of the quality of the process to ensure that the manufactured products meet the specifications. Considering the following assumptions: (i) that process must be under control and stable, and (ii) that the process data are independent and normally distributed [2], have been proposed several PCIs:

- Potential process capability index: $C_p = \frac{USL-LSL}{6\sigma}$
- Lower one sided capability index: $C_{pl} = \frac{\mu-LSL}{3\sigma}$
- Upper one sided capability index: $C_{pu} = \frac{USL-\mu}{3\sigma}$
- Real process capability index: $C_{pk} = min\{C_{pl}, C_{pu}\}$

where USL and LSL are the upper and lower specification limit respectively, μ is the mean of the process, and σ is the standard deviation of the process. Note that to processes with bilateral specifications (upper and lower limits) are applied the indices C_p and C_{pk}; whereas for processes with unilateral specifications (upper limit or lower limit) are considered the indices C_{pu} and C_{pl} as appropriate.

Most of the processes in the real world produce non-normal data [2]. The most commonly used methods for estimating PCIs associated with non-normal data are [3, 4, 5]:

- **Clements's Percentile Method:** It calculates the indices using a family of Pearson Curves. This method consists of estimating process capability indices as if they were normal but with two simple slight changes: Instead of using values of percentile points from normal distributions it takes them from non-normal distributions, and instead of taking the mean as the measure of central tendency it uses the median. In this case, the PCIs are defined as

$$C_p = \frac{USL - LSL}{P_{0.99865} - P_{0.00135}} \quad (1)$$

$$C_{pl} = \frac{P_{0.5} - LSL}{P_{0.5} - P_{0.00135}} \quad (2)$$

$$C_{pu} = \frac{USL - P_{0.5}}{P_{0.99865} - P_{0.5}} \quad (3)$$

  where P_q is the $q\text{-}th$ percentile value of non-normal distribution.

- **Burr's Percentile Method:** This is a change to Clements's Percentile Method by substituting the percentiles in the family of Pearson Distribution Curves with an appropriated Burr XII distribution.

- **Box-Cox Transformation Method:** It consists of an initial data transformation followed by the application of conventional methods to resulting data considered as normal.

Several contributions of some authors have been reported in technical literature for non-normal processes. Liu and Chen [6] develop a new estimation method called "Burr method for non-normal data", modifying Clements method and using the Burr XII distribution. Ahmad, Abdollahian and Zeephongseku [2] by establishing comparisons among those methods by using simulation techniques show that the method of the cumulative distribution function based on the Burr distribution, produces better estimates of process capability index. Additionally, they show that the method of Burr is better in accuracy than Clements's method.
Some of these researchers have used other approaches to calculate process capability indices. Abbasi [7] develops a

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neural network with a hidden layer to estimate index capability to non-normal processes, using as input variables the coefficients of skewness, and kurtosis, and the lower specification limit. Hosseinifard, Abbasi, Ahmad, and Abdollahian [8] use the transformation technique called "root transformation" in the estimation of process capability index when the data are not normal. The method is validated using simulated data from the Gamma, Beta and Weibull distributions. Leiva et al. [9] develop the theory related to the estimation and inference of process capability index, whose data come from the Birnbaum-Saunders distribution. In addition, their methodology is implemented in the R software.

Moreover, when output of process is not normally distributed, it is required the assumption of an empirical distribution of the data to analyze. Commonly used probability distributions are characterized by being positive asymmetric. The distributions used to describe the process outputs are: Generalized Pareto with 2 parameters, Weibull, Log-Normal and Beta distributions [3, 4, 10]. In the literature review was found that negative asymmetrical distributions have not been considered yet, omitting the proper study of processes that may have this behavior. Among the probability distributions with negative asymmetric behavior, the Skewed Normal distribution stands out.

The objective of this work is to develop capability indices for processes with non-normal data using Skewed Normal distribution covering this way the study of three possible expected processes behavior which are symmetrical, positive asymmetric and negative asymmetric. The importance and originality of this work is established as follows: According to the systematic literature review, there are no proposals for estimating process capability indices under Skewed Normal distribution, and the asymmetrical flexibility given by this distribution will render its worth by giving robust process capability indices and making easier the data fitting.

This paper consists of five sections, addressing first all related to the Skewed Normal distribution. The calculating of the PCIs using Skewed Normal distribution is described in Section III. A simulation study is presented in Section IV, and finally the conclusions are given in Section V.

II. Skewed Normal Distribution

A. Mathematical specification

The Skewed Normal distribution is just an extent of Normal distribution by adding a shape parameter that gives three possible distribution shapes: positive asymmetry, negative asymmetry and symmetry itself [11]. The probability density function associated to a random variable with a Skewed Normal distribution is as follows:

$$f(x) = \frac{1}{\omega\pi} e^{-\frac{(x-\xi)^2}{2\omega^2}} \int_{-\infty}^{\alpha\left(\frac{x-\xi}{\omega}\right)} e^{-\frac{t^2}{2}}\, dt \qquad (4)$$

where $\xi \in \mathbb{R}$ is a position parameter, $\omega \in \mathbb{R}^+$ is a scaling parameter and $\alpha \in \mathbb{R}$ is a shape parameter.
Several important fields of knowledge have intensively applied this statistical distribution like Risk Assessment and Capital Assignation commonly found in Actuarial Science, Insurance and Risk Management [12]. Nevertheless, this distribution has not been used in the capability analysis of industrial processes.

B. Estimation Methods

By using the classical method of maximum likelihood estimators for parameters involved in (4), it follows that the likelihood function is given by:

$$L(\xi,\omega,\alpha) = \frac{2^n}{\omega^n} \prod_{i=1}^{n} \phi\left(\frac{x_i-\xi}{\omega}\right) \Phi\left(\alpha\frac{x_i-\xi}{\omega}\right) \qquad (5)$$

where ϕ (.) and Φ (.) stand for probability density function and cumulative distribution function for standard normal, respectively.
The equations obtained by maximizing the likelihood function logarithm given in (5) with respect to ξ, ω and α, cannot solve analytically, and therefore require the use of numerical methods [13]. The function *selm* of package *sn* of R software allows obtain parameter estimates associated to a sample data from of the Skewed Normal distribution.

III. PCI Calculation Method Using Skewed Normal Distribution

To calculate the PCI of process data skewed normally distributed, an adaptation of the method of percentiles of Clements is proposal. In the flowchart (see Figure 1) is shown the following process of estimation of the PCIs, depending of asymmetric of data:

1. Input the process data.
2. Estimate parameters ξ, ω and α using *selm* function in R.
3. Develop a data related histogram to analyze data behavior and with the estimated α at hand establish if tendency of process is just symmetrical, asymmetrical positive or asymmetrical negative.
4. Based upon estimated α calculate corresponding percentiles using *qsn* function in R and then determine process capability indices accordingly.
    a) If the value of the estimated α is negative, $P_{0.00135}$ and $P_{0.5}$ are calculated and replaced in (2).
    b) If the value of α is zero then $P_{0.00135}$, $P_{0.5}$ and $P_{0.99865}$ are evaluated and replaced in $(1)-(3)$.
    c) If estimated α is positive $P_{0.5}$ and $P_{0.99865}$ are assessed and replaced in (3).

The corresponding flowchart for process capability indices calculation is presented in Figure 1:

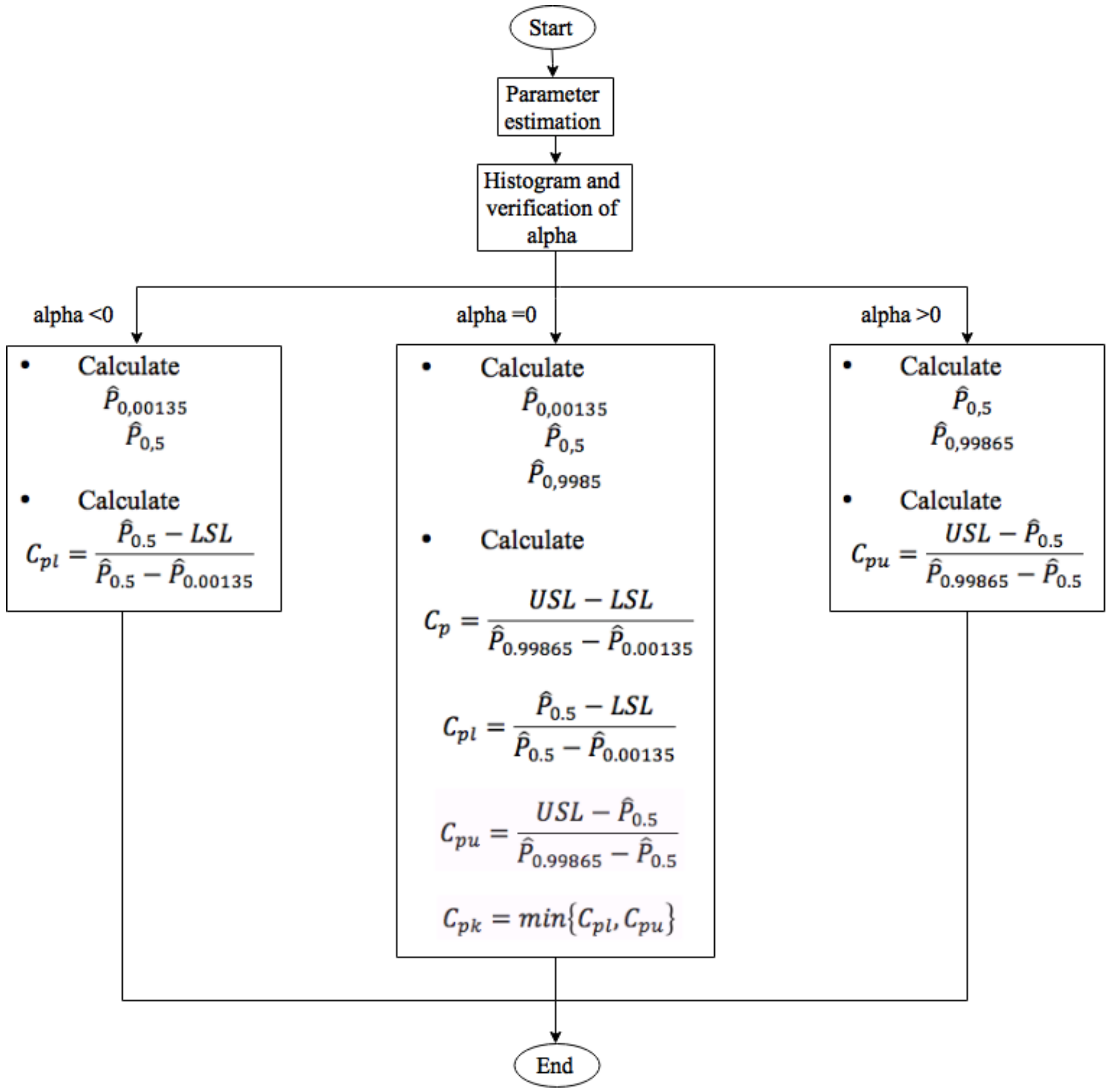

Figure 1. Flowchart for process capability indices calculation.

IV. Simulation Study

A. Comparison criteria

The simulation analysis is performed to investigate the effects of non-normally distributed data on process capability indices. In first place, indices calculation is accomplished using 30 samples of 500 size assessing their behavior in terms of different position parameters ξ and scaling parameters ω. Then, the comparison of estimated indices against real indices is performed, this time using again 30 samples but varying sizes to different values (50, 100, 200 and 500). Both procedures were performed as follows:

1. Set indices

   a) For the first case:
   $C_{pl}, C_{pu} = 1.5$

   b) For the second case:
   $C_{pl}, C_{pu} = \{0.5,1,1.5,2\}$.

2. Set parameters

   a) For the first case:
   $\xi = \{-2,0,2\}$,
   $\omega = \{0.5,1,1.5,2,2.5,3\}$,
   $\alpha = \{-3,-2,-1,0,1,2,3\}$.

   b) For the second case:
   $\xi = 2$,
   $\omega = 0.5$,
   $\alpha = \{-3,-2,-1,0,1,2,3\}$.

3. Calculate specification limits:

$$LSL = P_{0,5} - C_{pl}(P_{0,5} - P_{0,00135})$$
$$USL = C_{pu}\left(P_{0,99865} - P_{0,5}\right) + P_{0,5}$$

4. Generate skewed normal data with set parameters ξ, ω and α.

5. Estimate process capability indices as described in Section III (See Figure 1).

B. Results

The analysis on indices behavior is presented in the following figures:

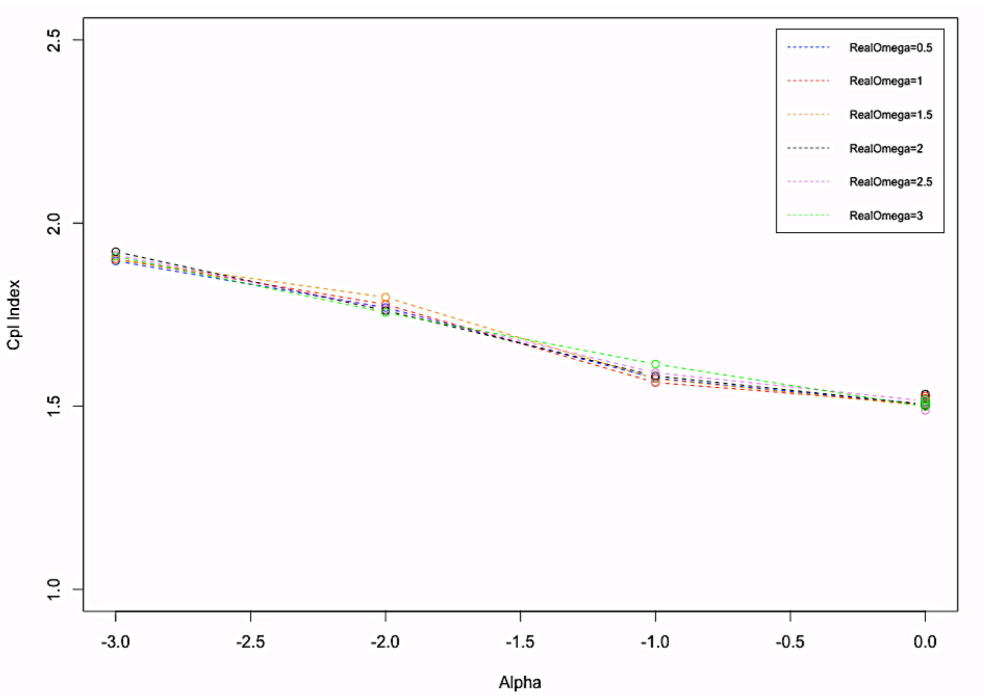

Figure 2. C_{pl} Index with mean $\xi = -2$, sample size $n = 500$ and various scaling factors ω.

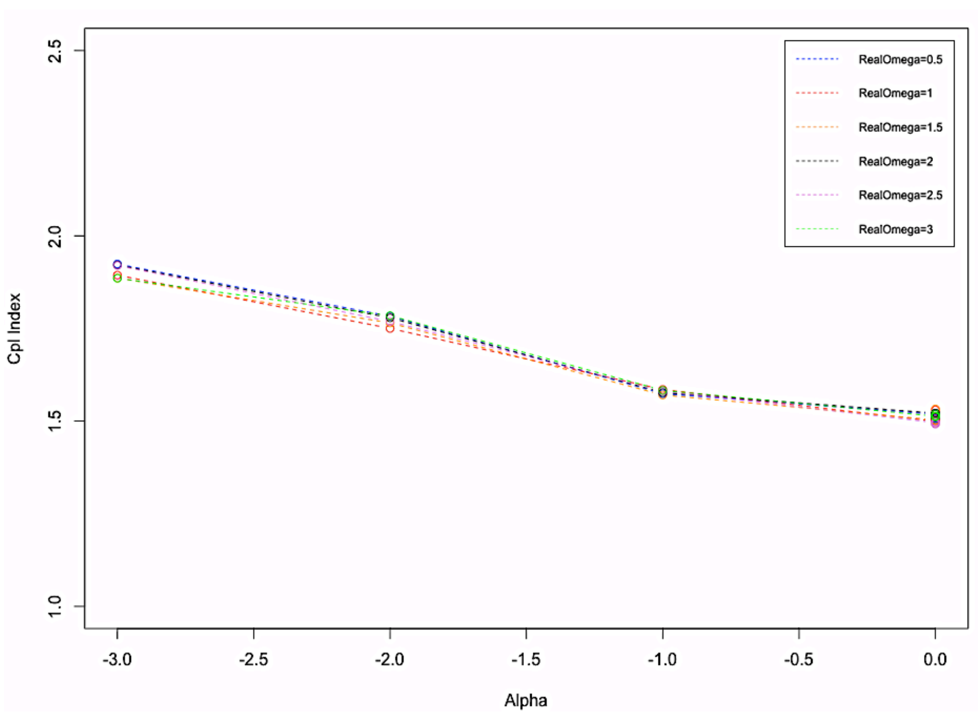

Figure 3. C_{pl} Index with mean $\xi = 0$, sample size $n = 500$ and various scaling factors ω.

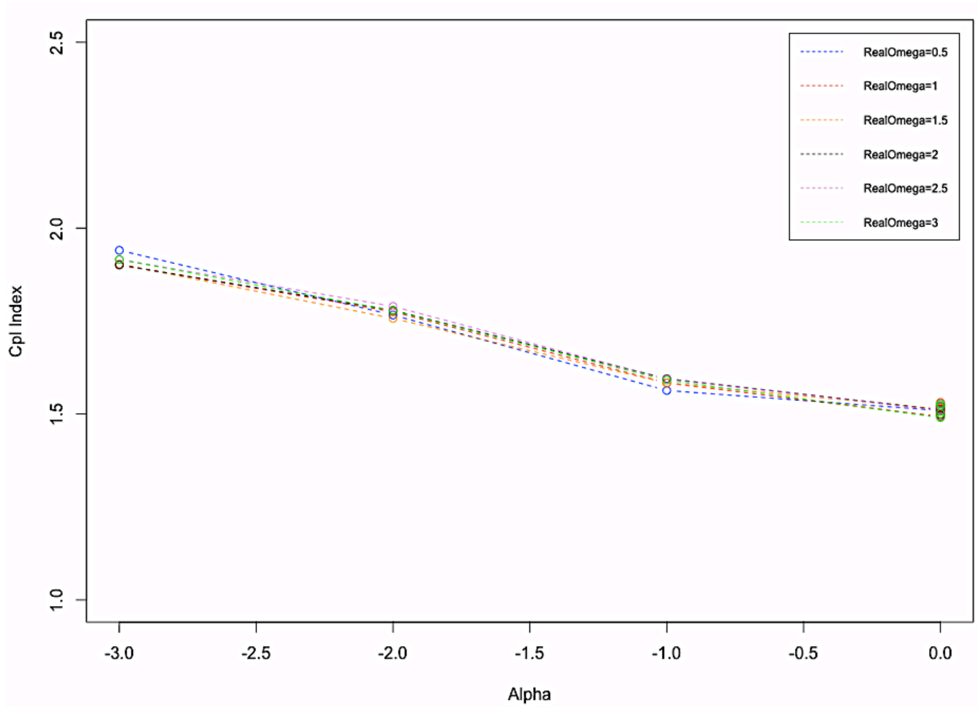

Figure 4. C_{pl} Index with mean $\xi = 2$, sample size $n = 500$ and various scaling factors ω.

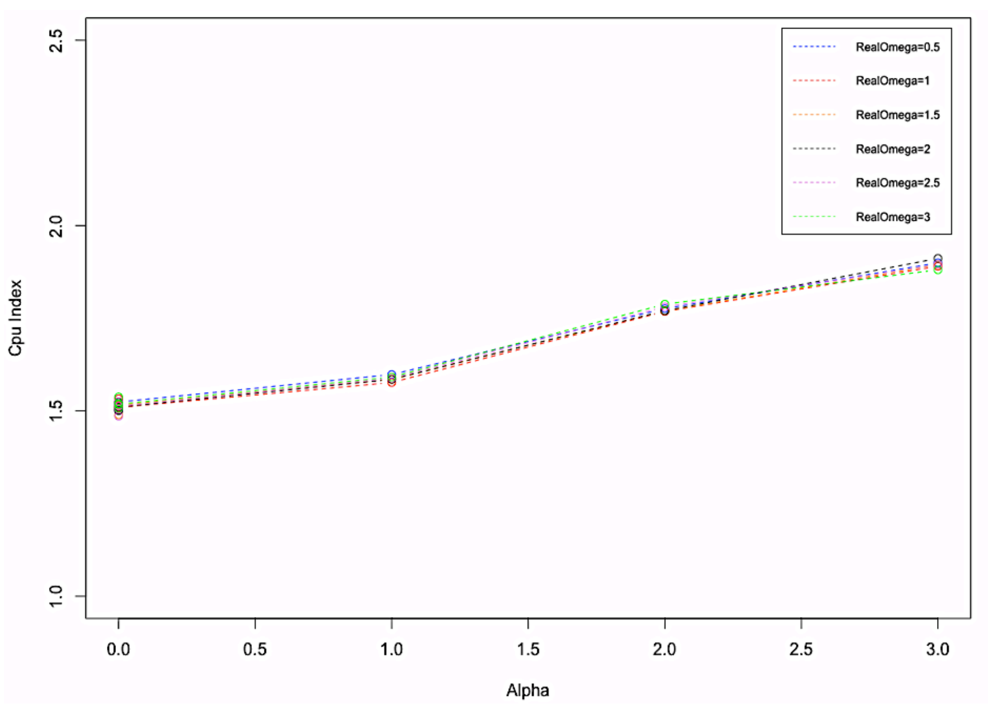

Figure 5. C_{pu} Index with mean $\xi = -2$, sample size $n = 500$ and various scaling factors ω.

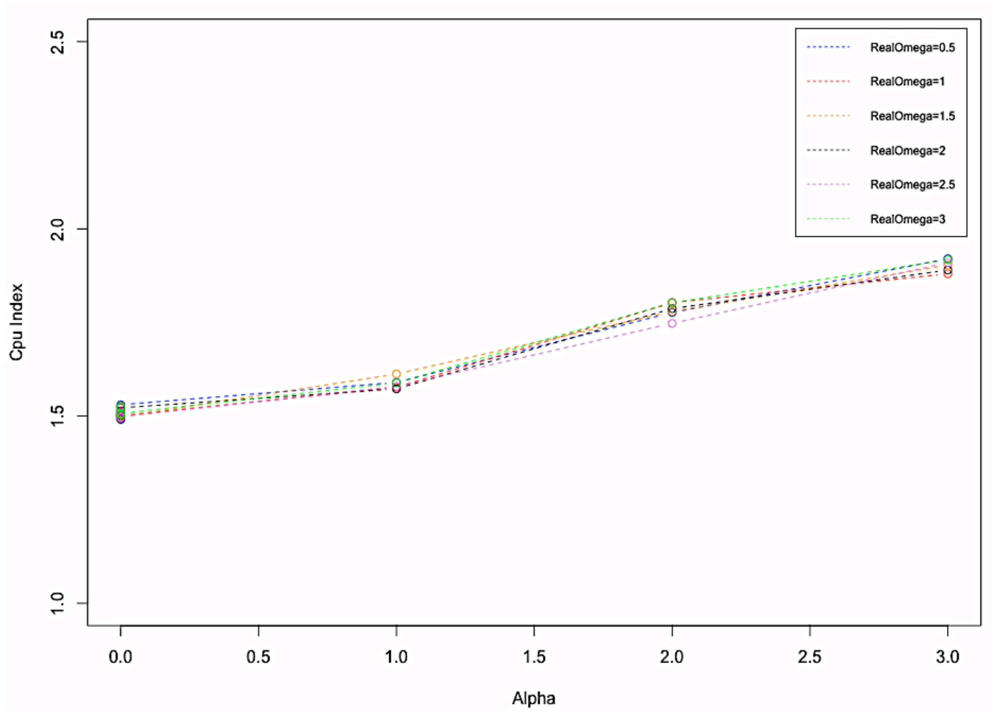

Figure 6. C_{pu} Index with mean $\xi = 0$,sample size $n = 500$ and various scaling factors ω.

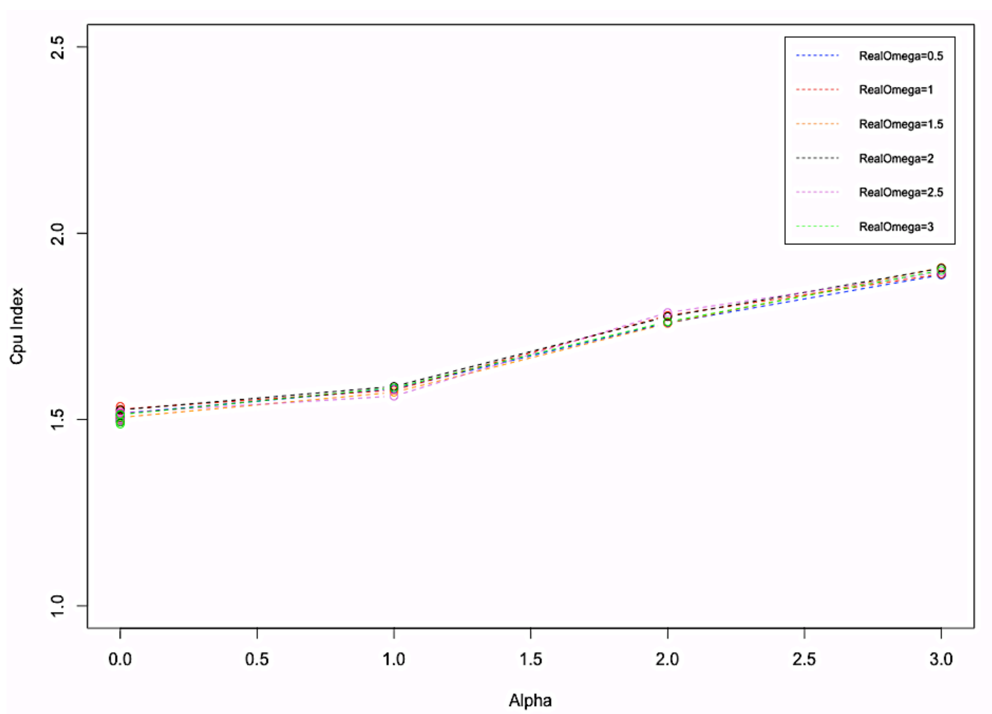

Figure 7. C_{pu} Index with mean $\xi = 2$, sample size $n = 500$ and various scaling factors ω.

To compare estimated indices against real indices, results are presented in both Figures 8, 9, 10 and 11 and Tables 1, 2 and 3.

		50		100		200		500	
α	**$Real$ C_{pl}**	**$Mean$**	**sd**	**$Mean$**	**sd**	**$Mean$**	**sd**	**$Mean$**	**sd**
-3	**0,5**	0,530	0,126	0,513	0,080	0,507	0,055	0,490	0,035
	1	1,302	0,238	1,201	0,125	1,235	0,096	1,201	0,051
	1,5	2,046	0,320	1,917	0,156	1,931	0,118	1,902	0,079
	2	2,606	0,271	2,622	0,225	2,588	0,169	2,623	0,119
-2	**0,5**	0,515	0,121	0,505	0,090	0,505	0,054	0,509	0,034
	1	1,260	0,177	1,147	0,108	1,137	0,089	1,130	0,040
	1,5	1,806	0,273	1,811	0,139	1,798	0,105	1,795	0,077
	2	2,600	0,363	2,399	0,189	2,437	0,140	2,417	0,089
-1	**0,5**	0,524	0,167	0,513	0,069	0,489	0,065	0,489	0,032
	1	1,063	0,154	1,072	0,109	1,054	0,081	1,049	0,050
	1,5	1,720	0,208	1,637	0,143	1,602	0,101	1,603	0,055
	2	2,255	0,210	2,182	0,187	2,134	0,107	2,142	0,078

Table 1. Mean and standard deviation for 30 samples in C_{pl} index calculation.

		50		100		200		500	
α	**$Real$ C_{pu}**	**$Mean$**	**sd**	**$Mean$**	**sd**	**$Mean$**	**sd**	**$Mean$**	**sd**
1	**0,5**	0,522	0,141	0,497	0,097	0,496	0,060	0,505	0,030
	1	1,083	0,186	1,094	0,098	1,042	0,087	1,042	0,048
	1,5	1,683	0,255	1,580	0,131	1,606	0,100	1,581	0,066
	2	2,283	0,362	2,192	0,206	2,170	0,131	2,122	0,080
2	**0,5**	0,502	0,125	0,533	0,111	0,519	0,078	0,489	0,034
	1	1,146	0,130	1,170	0,106	1,158	0,089	1,148	0,059
	1,5	1,906	0,284	1,794	0,153	1,781	0,126	1,776	0,070
	2	2,415	0,329	2,461	0,204	2,444	0,165	2,416	0,087
3	**0,5**	0,524	0,136	0,501	0,075	0,511	0,064	0,494	0,034
	1	1,209	0,178	1,165	0,112	1,203	0,083	1,193	0,056
	1,5	1,940	0,201	1,933	0,207	1,909	0,109	1,875	0,079
	2	2,712	0,316	2,649	0,218	2,617	0,149	2,590	0,101

Table 2. Mean and standard deviation for 30 samples in the C_{pu} index calculation.

		C_{pu}			C_{pl}			C_p			C_{pk}	
n	**$Real$ C_{pu}**	**$Mean$**	**sd**	**$Real$ C_{pl}**	**$Mean$**	**sd**	**$Real$ C_p**	**$Mean$**	**sd**	**$Real$ C_{pk}**	**$Mean$**	**sd**
50	0,5	0,557	0,147	0,5	0,532	0,123	0,50	0,542	0,063	0,5	0,452	0,069
	0,5	0,507	0,115	1	1,095	0,197	0,75	0,799	0,081	0,5	0,507	0,115
	0,5	0,543	0,112	1,5	1,638	0,205	1,00	1,090	0,123	0,5	0,543	0,112
	0,5	0,566	0,118	2	2,063	0,248	1,25	1,314	0,139	0,5	0,566	0,118
	1	1,076	0,174	0,5	0,568	0,120	0,75	0,821	0,088	0,5	0,568	0,120
	1	0,964	0,137	1	1,126	0,150	1,00	1,043	0,095	1,0	0,944	0,117
	1	1,098	0,175	1,5	1,556	0,196	1,25	1,326	0,143	1,0	1,097	0,174
	1	1,044	0,163	2	2,095	0,240	1,50	1,568	0,153	1,0	1,044	0,163
	1,5	1,648	0,221	0,5	0,550	0,124	1,00	1,098	0,125	0,5	0,550	0,124

	1,5	1,602	0,231	1	1,052	0,164	1,25	1,323	0,124	1,0	1,051	0,160
	1,5	1,645	0,226	1,5	1,574	0,171	1,50	1,608	0,147	1,5	1,501	0,141
	1,5	1,606	0,202	2	2,117	0,235	1,75	1,861	0,187	1,5	1,605	0,202
	2	2,186	0,237	0,5	0,551	0,115	1,25	1,369	0,122	0,5	0,551	0,115
	2	2,178	0,272	1	1,102	0,199	1,50	1,638	0,192	1,0	1,102	0,199
	2	2,084	0,234	1,5	1,551	0,157	1,75	1,816	0,162	1,5	1,550	0,156
	2	2,124	0,286	2	2,099	0,188	2,00	2,111	0,212	2,0	2,025	0,216
100	0,5	0,515	0,086	0,5	0,524	0,100	0,50	0,519	0,042	0,5	0,453	0,055
	0,5	0,513	0,083	1	1,080	0,148	0,75	0,795	0,074	0,5	0,513	0,083
	0,5	0,500	0,069	1,5	1,526	0,160	1,00	1,013	0,090	0,5	0,500	0,069
	0,5	0,531	0,077	2	2,042	0,156	1,25	1,287	0,078	0,5	0,531	0,077
	1	1,066	0,127	0,5	0,499	0,097	0,75	0,782	0,067	0,5	0,499	0,097
	1	1,012	0,126	1	1,043	0,093	1,00	1,027	0,073	1,0	0,966	0,064
	1	1,019	0,098	1,5	1,525	0,137	1,25	1,272	0,097	1,0	1,019	0,098
	1	1,071	0,148	2	2,041	0,157	1,50	1,557	0,129	1,0	1,071	0,148
	1,5	1,515	0,164	0,5	0,508	0,075	1,00	1,011	0,073	0,5	0,508	0,075
	1,5	1,514	0,137	1	1,075	0,117	1,25	1,295	0,099	1,0	1,075	0,117
	1,5	1,587	0,154	1,5	1,547	0,160	1,50	1,567	0,125	1,5	1,486	0,106
	1,5	1,557	0,139	2	2,069	0,173	1,75	1,812	0,114	1,5	1,557	0,138
	2	2,111	0,223	0,5	0,530	0,087	1,25	1,321	0,121	0,5	0,530	0,087
	2	2,044	0,177	1	1,047	0,163	1,50	1,547	0,139	1,0	1,047	0,163
	2	2,057	0,193	1,5	1,530	0,149	1,75	1,793	0,143	1,5	1,530	0,149
	2	2,033	0,146	2	2,108	0,232	2,00	2,070	0,171	2,0	1,998	0,153
200	0,5	0,518	0,053	0,5	0,479	0,046	0,50	0,499	0,021	0,5	0,460	0,034
	0,5	0,500	0,057	1	1,015	0,094	0,75	0,758	0,041	0,5	0,500	0,057
	0,5	0,524	0,058	1,5	1,524	0,102	1,00	1,024	0,059	0,5	0,524	0,058
	0,5	0,512	0,063	2	2,069	0,126	1,25	1,290	0,054	0,5	0,512	0,063
	1	1,049	0,060	0,5	0,507	0,051	0,75	0,778	0,032	0,5	0,507	0,051
	1	1,029	0,076	1	1,021	0,079	1,00	1,025	0,055	1,0	0,981	0,066
	1	1,022	0,068	1,5	1,556	0,087	1,25	1,289	0,062	1,0	1,022	0,068
	1	0,989	0,064	2	2,049	0,127	1,50	1,519	0,078	1,0	0,989	0,064
	1,5	1,567	0,128	0,5	0,502	0,055	1,00	1,034	0,058	0,5	0,502	0,055
	1,5	1,536	0,098	1	1,011	0,060	1,25	1,273	0,058	1,0	1,011	0,060
	1,5	1,548	0,089	1,5	1,580	0,101	1,50	1,564	0,078	1,5	1,517	0,067
	1,5	1,543	0,102	2	2,026	0,119	1,75	1,785	0,093	1,5	1,543	0,102
	2	2,028	0,087	0,5	0,508	0,052	1,25	1,268	0,051	0,5	0,508	0,052
	2	2,046	0,138	1	1,022	0,062	1,50	1,533	0,086	1,0	1,022	0,062
	2	2,032	0,099	1,5	1,535	0,076	1,75	1,783	0,064	1,5	1,535	0,076
	2	2,029	0,142	2	1,999	0,130	2,00	2,014	0,130	2,0	1,976	0,124
500	0,5	0,507	0,037	0,5	0,510	0,033	0,50	0,509	0,014	0,5	0,481	0,021
	0,5	0,499	0,040	1	1,000	0,043	0,75	0,750	0,020	0,5	0,499	0,040
	0,5	0,502	0,044	1,5	1,504	0,078	1,00	1,003	0,037	0,5	0,502	0,044
	0,5	0,509	0,043	2	2,000	0,055	1,25	1,255	0,035	0,5	0,509	0,043
	1	1,008	0,053	0,5	0,505	0,041	0,75	0,756	0,029	0,5	0,505	0,041

	1	0,991	0,034	1	1,020	0,053	1,00	1,006	0,030	1,0	0,977	0,028
	1	1,008	0,045	1,5	1,517	0,065	1,25	1,263	0,041	1,0	1,008	0,045
	1	1,006	0,043	2	2,008	0,077	1,50	1,507	0,047	1,0	1,006	0,043
	1,5	1,523	0,071	0,5	0,494	0,036	1,00	1,008	0,034	0,5	0,494	0,036
	1,5	1,499	0,059	1	0,997	0,040	1,25	1,248	0,036	1,0	0,997	0,040
	1,5	1,516	0,068	1,5	1,526	0,065	1,50	1,521	0,057	1,5	1,495	0,054
	1,5	1,510	0,068	2	2,011	0,069	1,75	1,760	0,061	1,5	1,510	0,068
	2	1,993	0,068	0,5	0,508	0,044	1,25	1,251	0,040	0,5	0,508	0,044
	2	2,002	0,066	1	0,996	0,046	1,50	1,499	0,041	1,0	0,996	0,046
	2	1,988	0,070	1,5	1,491	0,056	1,75	1,740	0,055	1,5	1,491	0,056
	2	1,997	0,086	2	2,003	0,064	2,00	2,000	0,066	2,0	1,972	0,070

Table 3. Mean and standard deviation for 30 samples in C_{pu}, C_{pl}, C_p and C_{pk} indices calculation.

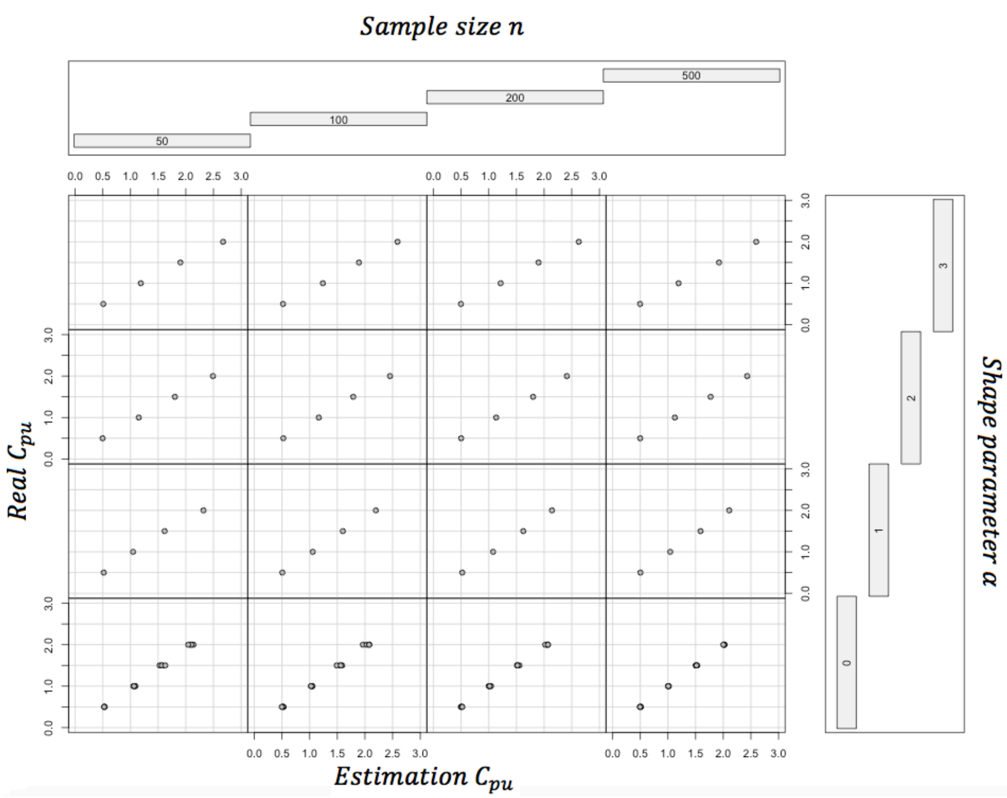

Figure 8. Comparison between estimated C_{pu} index and the real value.

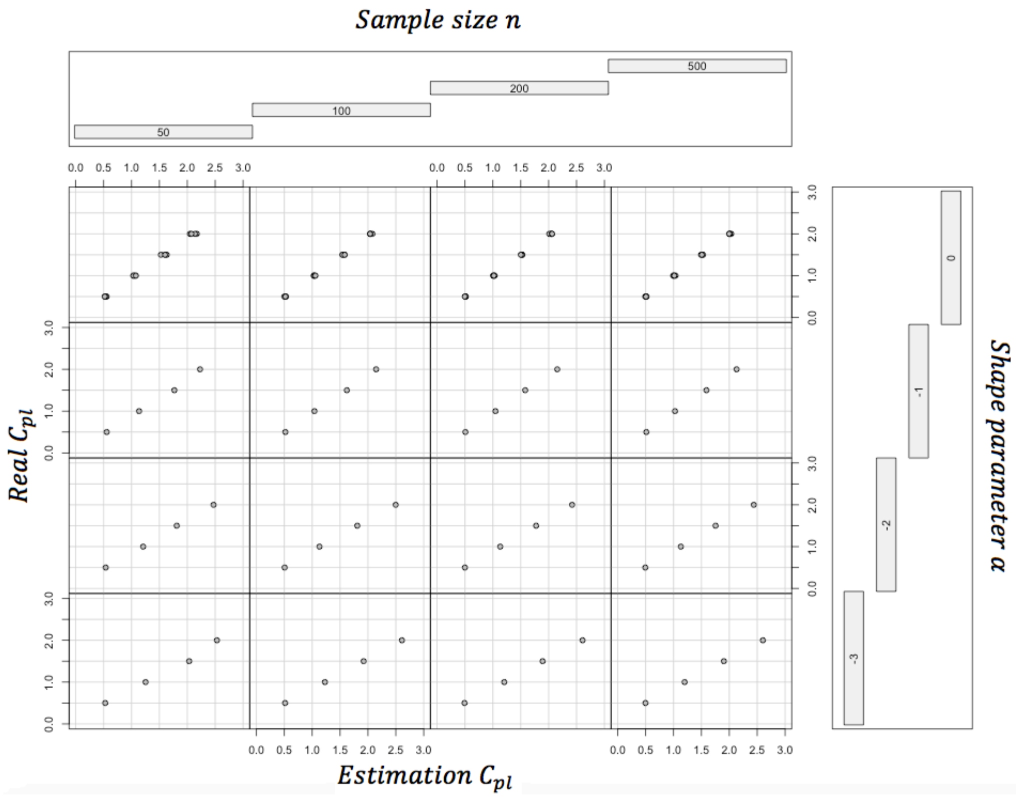

Figure 9. Comparison between estimated C_{pl} index and the real value.

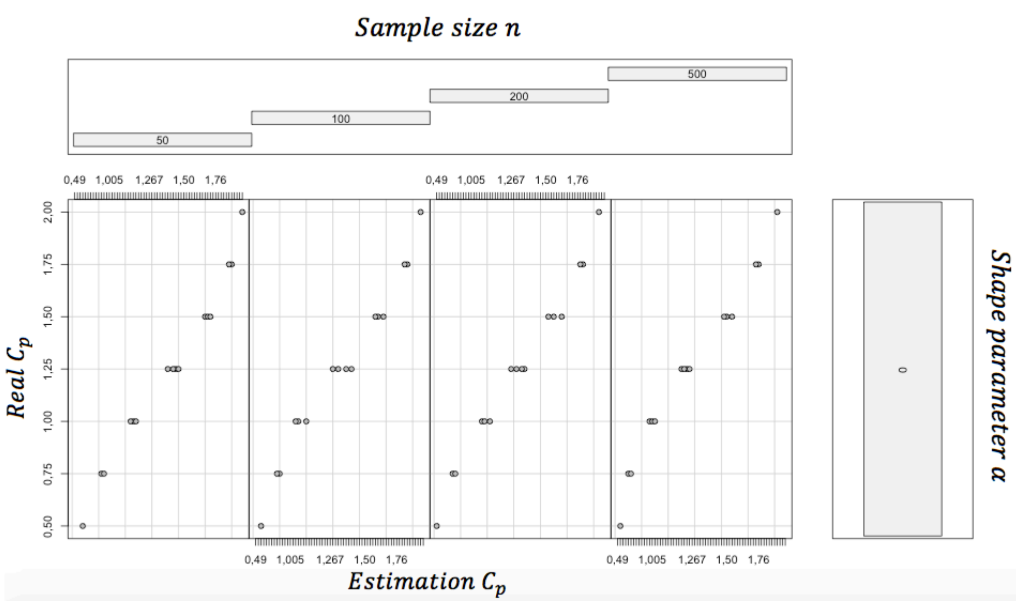

Figure 10. Comparison between estimated C_p index and real value.

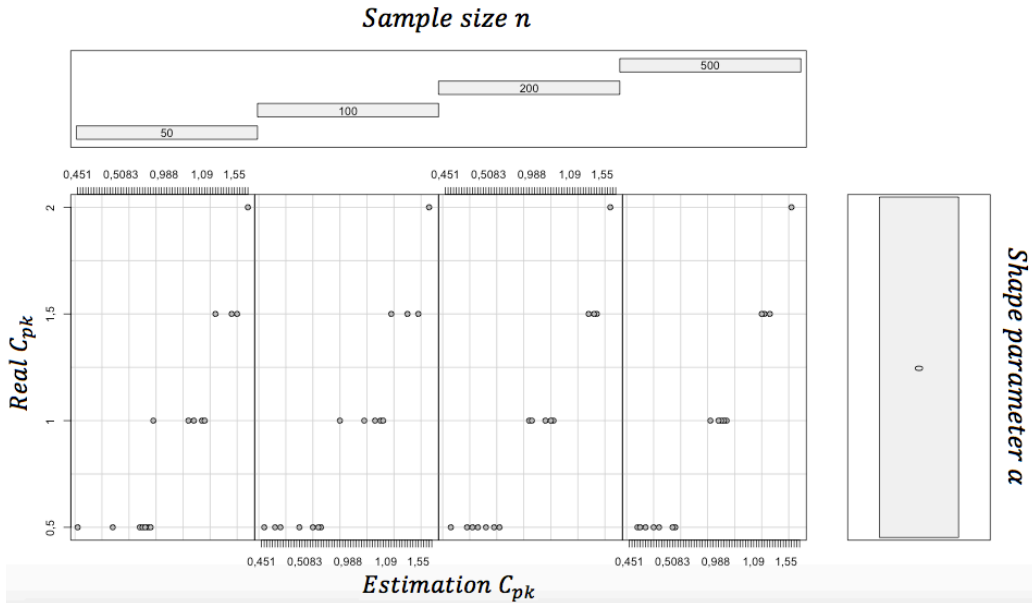

Figure 11. Comparison between estimated C_{pk} index and the real value.

C. Discussion

Based upon the results obtained in Section III it can be said that capability indices are consistent estimators because as the sample sizes increase the estimated indices tend to their real values and the standard deviations become smaller, as evidenced in Tables 1, 2 and 3. Besides that, it is remarkable that C_{pu} and C_{pl} indices get closer to real values as the shape parameter α is closer to zero because their behavior is related to normal distributed data.

The behavior of all indices is the expected as given by the trends to a straight line obtained and shown in Figures 8, 9, 10 and 11.

V. Conclusions

Real data associated to production processes do not always follow a normal distribution, even though most industries assume normal behavior. Real process data are flexible and are influenced by various external factors requiring considering process capability indices for asymmetric distributions, or giving wrong information otherwise and preventing industry of performing what is really needed.

Skewed Normal Distribution is an excellent tool to fit real process data into capability indices since it can contemplate the three associated behaviors regarding asymmetric distributions.

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