Numerical performance of some Wong-Zakai type approximations for stochastic differential equations

Jaime Alberto Londoño Londoño, Andrés Mauricio Villegas Ramirez

Abstract

A Wong-Zakai type numerical method for the strong solution of stochastic differential equations is introduced and developed. The main feature of the method is that it takes advantage of the well developed techniques for solution of ordinary differential equations. Focus is given to the evaluation of the numerical performance of the method.

Keywords: Stochastic differential equations; Numerical simulation; Ordinary differential equations.

1 Introduction

Ordinary differential equations (ODEs) are a common tool for modelling physical systems. Such models represent idealized versions of real systems, as they are purely deterministic. Stochastic differential equations (SDEs) are the instrument for building more realistic models, as they include the random elements. SDEs are used in many areas of applications, including investment finance, economics, insurance, signal processing and filtering, several fields of biology and physics, population dynamics and genetics.

Since there are very few SDEs for which exact analytical solutions are known, numerical techniques have to be used in the solution of SDEs. Unlike in the solution of ODEs, SDEs can be approximated in two senses: strong and weak. We refer to strong approximations when we want to approximate the trajectory of the solution. When we do not require the whole trajectory, but a function from the solution, for instance a moment, we talk about weak approximations. Some introductions to numerical methods for SDEs are given in [1, 6, 10]

A general Stratonovich SDE has the form

$$dX_s = b(X_s, s)ds + \sigma(X_s, s) \circ dW_s \qquad X_{t_0} = X_0 \qquad X_0 \in \mathbb{R}^n \tag{1}$$

where $b : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$ is a d-dimensional vector valued function, $\sigma : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^{n\times d}$ is $n \times d$-matrix valued function, and W_s is a d-dimensional Brownian motion. Commonly, b and σ are called the drift and the diffusion coefficient, respectively.

It its the goal of this paper to present a Wonk-Zakai type numerical method for the strong solution of SDEs, whose main feature is that it uses the well developed techniques of solution of ODEs.

The remainder of this paper is organized as follows. In section 2 the proposed numerical technique is described. In section 3 it is shown that proposed method might be reduced to the Milstein scheme. In section 4 a sufficient and necessary condition for the method to be exact is derived. In section 5 some implementation issues are discussed. Finally, in section 6 some numerical results are presented.

2 The solution method

We consider the class of Stratonovich stochastic differential equations where the diffusion coefficient is a function of the state variables. That is

$$dX_s = b(X_s, s)ds + \sigma(X_s) \circ dW_s \qquad X_{t_0} = X_0 \qquad X_0 \in \mathbb{R}^n \tag{2}$$

For this type of equations a Wong-Zakai type numerical solution method is proposed. Wong-Zakai type approximations were first introduced in [13].

Assume that we wish to approximate the solution to the above equations at points $0 = t_0 < t_1 < t_2 < \cdots < t_k = T$ of the interval $[t_0, T]$. Let X_j be the numerical approximation of X_{t_j}. For each subinterval $[t_j, t_{j+1}]$,$j = 0, 1, \ldots, k-1$, X_{j+1} shall be calculated as the solution at time t_{j+1} to the following ordinary differential equation initial value problem:

$$\frac{d\hat{X}_s}{ds} = b(\hat{X}_s, s) + \frac{1}{\Delta_j}\sigma(\hat{X}_s)\Delta W_j \qquad \hat{X}_{t_j} = X_j \qquad X_j \in \mathbb{R}^n \tag{3}$$

where $\Delta_j = t_{j+1} - t_j$ and $\Delta W_j = W_{t_{j+1}} - W_{t_j}$ are the discrete time approximations of ds and dW_s, respectively.

Equation (3) may be rewritten in integral form to obtain the following numerical scheme

$$X_{j+1} = X_j + \int_{t_j}^{t_{j+1}} b(X_s, s)ds + \int_{t_j}^{t_{j+1}} \frac{1}{\Delta_j}\sigma(X_s)\Delta W_j ds \tag{4}$$

3 Reduction to the Milstein scheme

We shall now make some remarks on the 1-dimensional case $n = d = 1$.

The Milstein scheme is one of the simplest discrete time approximations derived from stochastic Taylor expansions. In the 1-dimensional case this scheme has the form

$$X_{j+1} = X_j + b(X_j, t_j)\Delta_j + \sigma(X_j, t_j)\Delta W_j + \frac{1}{2}\sigma(X_j, t_j)\sigma'(X_j, t_j)(\Delta W_j)^2 \tag{5}$$

where $\sigma'(x,t) = \dfrac{\partial \sigma}{\partial x}$. Under the assumption that b is once and σ twice continuously differentiable it can be shown that the Milstein scheme has the order of strong convergence $\gamma = 1.0$. A detailed explanation of the Milstein scheme is found in [6].

We shall see that that scheme given in (4) becomes the Milstein scheme if some particular methods are used to approximate the Riemann-Stieltjes integrals

$$\int_{t_j}^{t_{j+1}} b(X_s, s)ds$$

and

$$\int_{t_j}^{t_{j+1}} \frac{1}{\Delta_j}\sigma(X_s)\Delta W_j ds$$

The first integral may be approximated using the Euler method, which gives

$$\int_{t_j}^{t_{j+1}} b(X_s, s)ds \approx b(X_j, t_j)(t_{j+1} - t_j) = b(X_j, t_j)\Delta_j \tag{6}$$

In the other hand, the second integral may be approximated using the 2nd order truncated Taylor method, which uses the approximation

$$\int_{t_j}^{t_{j+1}} f(X_s, s)ds \approx f(X_j, t_j)\Delta_j + \frac{1}{2} f(X_j, t_j) f'(X_j, t_j)(\Delta_j)^2$$

Taking $f(x, t) = (\Delta_j)^{-1}\sigma(x)\Delta W_j$ we have

$$\int_{t_j}^{t_{j+1}} \frac{1}{\Delta_j}\sigma(X_s)\Delta W_j ds \approx \sigma(X_j)\Delta W_j + \frac{1}{2}\sigma(X_j)\sigma'(X_j)(\Delta W_j)^2 \tag{7}$$

Replacing (6) and (7) in (4) we obtain

$$X_{j+1} = X_j + b(X_j, t_j)\Delta_j + \sigma(X_j, t_j)\Delta W_j + \frac{1}{2}\sigma(X_j)\sigma'(X_j)(\Delta W_j)^2$$

which is the Milstein scheme for $\sigma(x, t) \equiv \sigma(x)$

4 Particular cases where the solution is exact

We shall find a sufficient and necessary condition on b and σ, for which the proposed numerical scheme yields the exact solution to (8), given that we are able to solve analytically the ODEs problems given by (3). We consider the class of scalar stochastic differential equation given by

$$dX_s = b(X_s)ds + \sigma(X_s) \circ dW_s \tag{8}$$

From equation (3) we may approximate X_t as the solution at time t to,

$$\frac{d\hat{X}_s}{ds} = b(\hat{X}_s) + \frac{1}{t}\sigma(\hat{X}_s)W_t \tag{9}$$

with initial value X_0 at $s = 0$. The successive solution of (9) for each $t \in [0, T)$ defines a function $u(t, x) \in C^2([0, T) \times \mathbb{R})$ such that $u(t, W_t)$ is the approximation to X_t.

Suppose that the numerical method produces the exact solution to (8), i.e., $X_t = u(t, W_t)$. Since Stratonovich integrals follow the classical chain rule formula, we conclude that

$$dX_s = \frac{\partial u}{\partial t}(s, W_s)ds + \frac{\partial u}{\partial x}(s, W_s) \circ dW_s$$

Since X_t is also the solution to (8), we see that

$$\frac{\partial u}{\partial t}(t, x) = b(u(t, x)) \tag{10}$$

and

$$\frac{\partial u}{\partial x}(t, x) = \sigma(u(t, x)) \tag{11}$$

We obtain from (10) and (11) the identities

$$\frac{\partial^2 u}{\partial x \partial t}(t, x) = b'(u(t, x))\sigma(u(t, x)) \tag{12}$$

and

$$\frac{\partial^2 u}{\partial t \partial x}(t, x) = \sigma'(u(t, x))b(u(t, x)) \tag{13}$$

where $f'(x)$ denotes $\frac{df}{dx}(x)$. Since $u(t, x) \in C^2([0, T) \times \mathbb{R})$ we have

$$\frac{\partial^2 u}{\partial x \partial t}(t,x) = \frac{\partial^2 u}{\partial t \partial x}(t,x)$$

Hence

$$b'(u(t,x))\sigma(u(t,x)) - \sigma'(u(t,x))b(u(t,x)) = 0$$

Notice that for this equation to be satisfied the functions b and σ must be such that

$$\frac{d}{dx}\left(\frac{b(x)}{\sigma(x)}\right) = 0$$

This implies that $\frac{b(x)}{\sigma(x)} = \alpha$, where α is an arbitrary constant. Thus

$$b(x) = \alpha\sigma(x) \tag{14}$$

This is a necessary condition for the numerical method to produce the exact solution to (8). We shall now see that (14) is also sufficient. We shall then prove that for the class of Stratonovich differential equations given by

$$dX_s = \alpha f(X_s)ds + f(X_s) \circ dW_s \tag{15}$$

with initial value X_0 at $t = 0$, the numerical method is exact. In [6] it is proved that the general solution to (15) is

$$X_t = h^{-1}(\alpha t + W_t + h(X_0)) \tag{16}$$

where h is given by

$$h(x) = \int^x \frac{ds}{f(s)}$$

For this particular type of stochastic differential equations, approximation (9) becomes

$$\frac{d\hat{X}_s}{ds} = \alpha f(\hat{X}_s) + \frac{1}{t} f(\hat{X}_s)W_t \tag{17}$$

You can use the principle of separation of variables to solve this equation at t:

$$\begin{aligned}
\frac{d\hat{X}_s}{f(\hat{X}_s)} &= \left(\alpha + \frac{W_t}{t}\right) ds \\
\int_{\hat{X}_0}^{\hat{X}_t} \frac{d\hat{X}_s}{f(\hat{X}_s)} &= \int_0^t \left(\alpha + \frac{W_t}{t}\right) ds \\
h(\hat{X}_t) - h(\hat{X}_0) &= \alpha t + W_t
\end{aligned}$$

Since $\hat{X}_0 = X_0$ we obtain

$$\hat{X}_t = h^{-1}(\alpha t + W_t + h(X_0)) \tag{18}$$

which is precisely (16), the solution to (15).

We have proved that condition (14) is sufficient and necessary for the proposed numerical method to produce the exact solution to (8). Geometric Brownian motions are an important type of processes which satisfies this condition.

5 Implementation issues

We shall now focus on the computer implementation of the described numerical method. First, we will rewrite equation (3) into a form that may be easier to implement.

Recall that the increments $W_t - W_s$ of a d-dimensional Brownian motion are $N(\mathbf{0}, (t-s)I)$ Gaussian distributed for $0 \leq s < t$, where I is the d-dimensional identity matrix. Thus, ΔW_j is $N(\mathbf{0}, \Delta_j I)$ Gaussian distributed. Then in (3), we can replace ΔW_j with $\sqrt{\Delta_j} Z$, to obtain

$$\frac{d\hat{X}_s}{ds} = b(\hat{X}_s, s) + \frac{1}{\sqrt{\Delta_j}} \sigma(\hat{X}_s) Z \qquad \hat{X}_{t_j} = X_j \qquad X_j \in \mathbb{R}^n \tag{19}$$

where Z is a Gaussian distributed random variable with mean $\mathbf{0}$ and covariance the d-dimensional identity matrix.

From equation (19), it becomes clear that the implementation of the numerical technique requires the generation of Gaussian distributed random variables and the solution of ODEs. Particularly, it is necessary to solve ODE initial value problems.

An ODE initial value problem can be written as

$$\frac{dX_t}{dt} = f(X_t, t), \qquad X_{t_0} = X_0, \qquad X_t \in \mathbb{R}^N \tag{20}$$

When solving ODE problems almost always numerical techniques must be used, since the available analytical techniques are not powerful enough to solve any ODE problem except the simplest. However, numerical methods for ODEs are rather more developed and available than numerical methods for SDEs.

ODE problems are generally divided into two categories, stiff and nonstiff problems. A proposed definition for stiffness is found in [11]. Generally a stiff problem is harder to solve than a nonstiff one. Therefore, the numerical methods for solving stiff equations are different from those for solving nonstiff equations. The methods used for solving nonstiff ODEs are based on the Runge-Kutta, Adams, or extrapolation methods. These methods are usually not suitable for solving stiff problems. The interested reader may refer to [5] for a survey on numerical methods for ODEs.

All the hard work of the proposed method relies on the numerical method used to solve the ODE initial value problems stated in (19). Therefore, the accuracy of the described numerical method depends on the accuracy of the ODE initial value problem solver. For instance, if the Euler method is used as the ODE solver, the propose method becomes the well known Euler-Maruyama method, and has thus strong order of convergence $\gamma = 0.5$.

For the numerical experiments we ran, we used CVODE as ODE solver. CVODE is a solver, written in C, for stiff and nonstiff initial value problems for systems of ordinary differential equations. The underlying integration methods used in CVODE are variable-coefficient forms of the Adams and BDF (Backward Differentiation formula) methods, for nonstiff and stiff problems, respectively. The interested reader may refer to [3] for a detailed presentation of CVODE and may download it from *http://www.netlib.org/ode/cvode.tar.gz.*

6 Numerical performance

In this section we will examine the numerical performance of the proposed numerical technique. Thus, we report numerical results for eighth test equations.

For the first four equations we shall focus on the accuracy of the algorithm. To measure the accuracy of the algorithm we use the mean of the absolute error $M(h)$ and the strong convergence rate R_1, defined by

$$M(h) = \frac{1}{N}\sum_{i=0}^{N} |X_k^{(i)} - X_{t_k}^{(i)}| \qquad R_1 = \frac{M(h)}{h}$$

where $X_k^{(i)}$ is the numerical approximation to $X_{t_k}^{(i)}$.

For each test equation we carried out a numerical simulation to get the 90% confidence intervals

$$[M(h) - \Delta M(h), M(h) + \Delta M(h)]$$

for the mean of the absolute error $M(h)$. In order to do this , 20 batches, each with 100 trajectories, were run.

In the other hand, for the last four test equations we shall mainly be concerned with the qualitative behavior of the simulated paths.

Test equation 1: This test equation is a one-dimensional nonlinear problem, taken from [12]. The nonlinear Stratonovich equation is given by,

$$dX_s = -\alpha(1 - X_s^2)ds + \beta(1 - X_s^2) \circ dW_s, \qquad X_0 = 0.5, \qquad t \in [0, 2] \tag{21}$$

with $\alpha = -1$ and $\beta = 1$. The exact solution of this equation is

$$X_t = \frac{(1 + X_0)\exp(-2\alpha t + 2\beta W_t)) + X_0 - 1}{(1 + X_0)\exp(-2\alpha t + 2\beta W_t) - X_0 + 1}$$

Notice that (21) is a particular case of (15) and thus the numerical scheme should produce the exact solution. Table 1 presents the simulation results for the first test equation. For step size $h = 1$ the mean absolute error is of order 1e-11 which is in contrast with the mean absolute error of order 0.01 for step size $h = 2^{-10}$ reported in [12], where Runge-Kutta type methods were used.

Table 1: Error and convergence rate of (21)

h	$M(h)$	R_1	$\Delta M(h)$
1	4.21e-11	4.21e-11	6.43e-12

Test equation 2: The second test equation is a nonlinear SDE given by,

$$dX_s = a(1 + X_s^2) \circ dW_s \qquad X_0 = 1 \qquad t \in [0, 1] \tag{22}$$

The exact solution is [2]

$$X_t = \tan(aW_t + \arctan X_0)$$

Notice that (22) is equation (15) with $f(x) = a(1 + x^2)$ and $\alpha = 0$. Table 2 gives means of the absolute error $M(h)$ and the radius Δh of the confidence interval, for time steps $h = 1$ and a

Table 2: Mean of the absolute error for (22)

a	$M(h)$	$\Delta M(h)$
0.15	1.05471e-12	1.05471e-12
0.2	1.88595e-11	1.88595e-11
0.25	2.32104e-08	2.32104e-08

ranging from 0.15 to 0.25. The mean of the absolute errors is of very small order, which confirms the results obtained in section 4.

Test equation 3: The third test equation, taken from [7], is a linear two-dimensional system, whose Stratonovich form is,

$$dY_t^1 = (-\frac{1}{4}\sigma^2(Y_t^1 + Y_t^2) - \frac{1}{4}\rho^2(Y_t^1 - Y_t^2))dt + \frac{1}{2}\sigma(Y_t^1 + Y_t^1) \circ dW_t^1 + \frac{1}{2}\rho(Y_t^1 - Y_t^2) \circ dW_t^2 \quad (23)$$

$$dY_t^2 = (-\frac{1}{4}\sigma^2(Y_t^1 + Y_t^2) + \frac{1}{4}\rho^2(Y_t^1 - Y_t^2))dt + \frac{1}{2}\sigma(Y_t^1 + Y_t^2) \circ dW_t^1 - \frac{1}{2}\rho(Y_t^1 - Y_t^2) \circ dW_t^2$$

$$Y_0 = [1, 0]^T \qquad t \in [0, 2] \quad (24)$$

The exact solution of this equation is

$$Y_t^1 = \frac{(X_t^1 + X_t^2)}{2}, \qquad Y_t^1 = \frac{(X_t^1 - X_t^2)}{2}$$

where

$$X_t^1 = (Y_0^1 + Y_0^2)\exp(-\frac{\sigma^2}{2}t + \sigma W_t^1) \quad \text{and} \quad X_t^2 = (Y_0^1 - Y_0^2)\exp(-\frac{\rho^2}{2}t + \rho W_t^1)$$

Table 3: Mean of the absolute error of (23) with $\rho = 1$ and $h = 2$. M_1 is for Y_t^1 and M_2 for Y_t^2

σ	$M_1(h)$	$\Delta M_1(h)$	$M_2(h)$	$\Delta M_2(h)$
2	9.31181e-10	5.85832e-10	9.06894e-10	5.86207e-10
4	6.8013e-11	3.22031e-11	6.19549e-11	3.21785e-11
6	4.91244e-10	4.85428e-11	4.88165e-10	4.88422e-11
8	8.33034e-10	1.43609e-10	8.32726e-10	1.43649e-10
10	1.04796e-09	1.65114e-10	1.04779e-09	1.65181e-10

Table 3 gives means of the absolute error for each of the two variables with $h = 2$, $\sigma \in [1, 10]$ and $\rho = 1$. The mean of the absolute errors is of a very small order, which suggests that the solution method returns the exact solution of the equation. This is in contrast with the numerical instabilities produced by the Euler approximation, as reported in [7].

Test equation 4: This test equation is a 2-dimensional linear SDE system, presented in [2], whose Stratonovich form is given by

$$dX_t = \left(U - \frac{1}{2}V^2\right)X_t dt + VX_t \circ dW_t, \qquad X_0 = [1 \quad 0.5]^T, \qquad t \in [0, 1] \quad (25)$$

where U and V are matrices

$$U = \begin{bmatrix} -u & u \\ u & -u \end{bmatrix} \qquad V = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$$

The exact solution of this equation is

$$X_t = P \begin{bmatrix} \exp(p^+(t)) & 0 \\ 0 & \exp(p^+(t)) \end{bmatrix} P^{-1} X_0, \qquad P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

where $p^{\pm}(t) = (-u - \frac{1}{2}v^2 \pm u)t + vW_t$ and $P = P^{-1}$

The stiffness of this system increases quadratically in terms of v. Table 4 gives the mean of the absolute error M of the numerical solution of (25) with $u = 5$, v ranging from 1 to 5 and stepsize $h = 1$. As v increases, the error magnitude does not increase and the stability of the solutions is maintained. This is in contrast with the poor stability properties of some Runge-Kutta type methods reported in [2] for $v \geq 3$. The great accuracy obtained suggests that the method converges exactly for this test equation.

Table 4: Error and convergence rate of (25) with $u = 5$ and $h = 1$, M_1 is for X_t^1 and M_2 for X_t^2

v	$M_1(h)$	$\Delta M_1(h)$	$M_2(h)$	$\Delta M_2(h)$
1	4.39168e-11	8.31936e-12	3.5004e-11	8.00839e-12
2	4.33394e-10	4.8326e-10	4.23236e-10	4.83385e-10
3	7.51537e-10	8.06515e-10	7.46363e-10	8.06445e-10
4	6.99382e-10	1.02481e-09	6.97279e-10	1.02468e-09
5	1.17071e-10	1.60928e-10	1.16575e-10	1.60903e-10

Test equation 5: This test equation corresponds to the interest rate model of Cox-Ingersoll-Ross (CIR) [4] for stochastic interest rates, whose Stratonovich form is

$$dX_t = [a + \frac{1}{4}\sigma^2 + bX_t]dt + \sigma\sqrt{X_t} \circ dW_t, \qquad X_0 >= 0 \tag{26}$$

where $a \geq 0$, $b \in \mathbb{R}$ and $\sigma > 0$ are parameters. It is known that the solutions are nonnegative. However, as well as some well known numerical techniques, the proposed numerical scheme produces trajectories which take negative values (see Figure 1). A nonnegative preserving numerical algorithm for the solution of (26) is presented in [9].

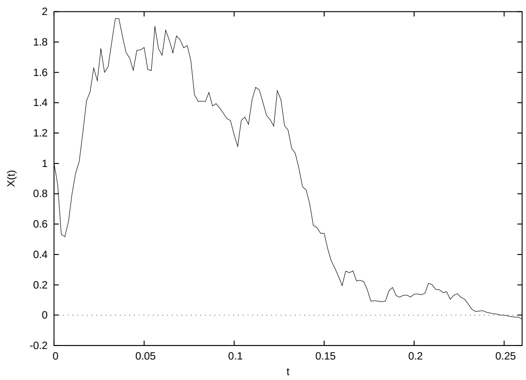

Figure 1: A sample path of (26) for $a = b = 1$, $\sigma = 2$, $h = 0.002$ and $X(0) = 1$

Test equation 6: The sixth example is the phased-locked loop (PLL), from the filering theory [1]. The PLL is an important FM demodulator, whose model is

$$dy_1 = -((c)^{1/3} y_1 + \sin y_2)dt + \sqrt{c} \circ dW_t^1 - \sqrt{c} \circ dW_t^2 \tag{27}$$
$$dy_2 = -(\frac{1}{2} y_1 - \sin y_2)dt + \sqrt{c} \circ dW_t^2$$

The deterministic model has stable equilibria at $y_1 = 0$, $y_2 = 2\pi n (n = 0, \pm 1, \dots)$. Any trajectory that begins in the domain of attraction of a stable equilibria will not leave this domain. However, in the stochastic model, small perturbations can cause a crossing of this domain. This is known as the phenomenon of slipping cycles. For small c slipping cycles are infrequent, but for big c they become more frequent. Figure 2 shows two trajectories produced by the proposed method. The numerical solution conserves the qualitative properties of the exact solution.

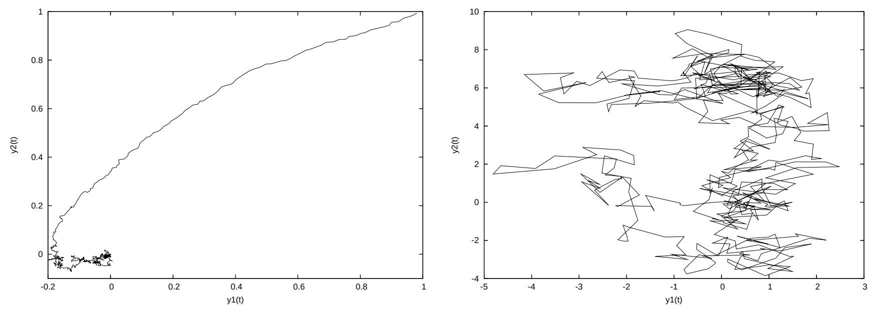

Figure 2: Phase plane of the PLL model with $c = 0.001$(left) and $c = 10$ (right)

Test equation 7: The seventh test equation is the stochastic duffing equation

$$dy_t = (\mu t y_t - y_t^3)dt + b \circ dW_t \tag{28}$$

It is important to study the relationship between μ and b. For example if $b \gg \mu$ the paths move back and forth between $\pm\sqrt{\mu t}$ before settling down on one branch [1]. Figure 3 plots the numerical solution obtained with $h = 0.01$, $\mu = 0.03$ and $b = 0.25, 0.75$. The solutions follow the previously describe characteristics.

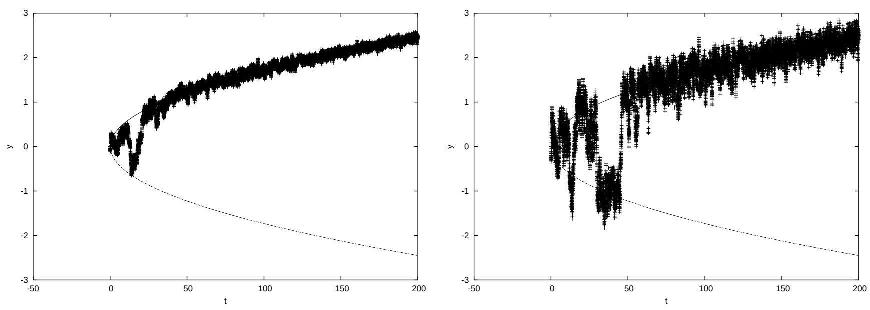

Figure 3: Plots y vs. t for the stochastic duffing problem. Left: $\mu = 0.03$, $b = 0.25$. Right $\mu = 0.03$, $b = 0.75$

Test equation 8: The last test equation is the Kubo oscillator whose Stratonovich form is

$$dX_t^1 = -aX_t^2 dt - \sigma X_t^2 \circ dW_t \qquad X_0^1 = x^1 \tag{29}$$
$$dX_t^2 = aX_t^1 dt + \sigma X_t^1 \circ dW_t \qquad X_0^2 = x^2$$

where a and σ are costants, and W_t is a one-dimensional Brownian motion. The quantity $H(x^1, x^2) = (x^1)^2 + (x^2)^2$ is conservative for the system; i.e,

$$H(X_t^1, X_t^2) = H(x^1, x^2) \quad \text{for } t \geq 1 \tag{30}$$

Therefore, the phase flow of this system preserves symplectic structure and its phase trajectory belongs to the circle with center at the origin and with radius $\sqrt{H(x^1, x^2)}$. For a discussion on stochastic systems preserving symplectic structure and symplectic numerical methods for such systems see [8].

We are interested in observing if the proposed numerical method preserves the conservative property (30). Figure 4 presents a sample phase trajectory of (29) with initial conditions $x^1 = 1$, and $x^2 = 0$. For these initial conditions the exact phase trajectory belongs to the unit circle with center at the origin. The trajectory obtained does not conserve $H(x^1, x^2)$, but stays close to the unit circle.

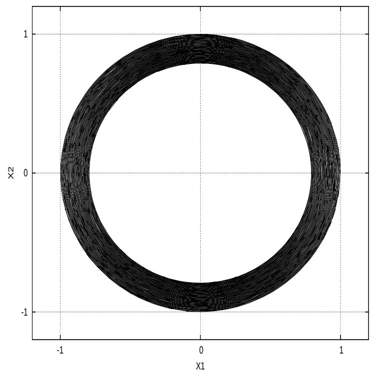

Figure 4: A sample phase trajectory of (29) with $X_0^1 = 1$, $X_0^2 = 0$, for $a = 2$, $\sigma = 0.3$, $h = 0.02$ on the time interval $[0, 200]$

7 Conclusions

In this paper, we have presented a Wong-Zakai type numerical method for the solution of stochastic differential equations. The main feature of the this method is that it uses the rather well developed and available numerical techniques for the simulation of ordinary differential equations. This allows one to implement it in any computer package that provides an ordinary differential equation solver.

We have also found a particular class of SDE for which the method produces the exact solution. The performance of the method was tested with several 1-dimensional and 2-dimensional equations. The results from the test problems suggested that the method produces very accurate approximations, and that it almost always keeps the qualitative behavior of the solution paths.

It still remains to be investigated the general strong convergence rate of the method and the extension of the method for general SDE where the diffusion term is a function of state and time.

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